Mathematical Reviews

Vol. 3, No. 2 February, 1942 pp. 33–64

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Published monthly by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

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THE MATHEMATICAL ASSOCIATION OF AMERICA
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Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

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Mathematical Reviews

Vol. 3, No. 2 FEBRUARY, 1942 Pages 33-64

THEORY OF GROUPS

Fokker, A. D. Les phénomènes propres des milieux cristallins. I. Physica 7, 385–412 (1940). [MF 4361]

An interesting attempt to derive the results of the group theoretical treatment of linear problems without explicit use of the theory of group representations. The author gives group tables for most crystallographic point groups and arrives, in the case of Abelian groups, at results which are equivalent to those of the usual theory. His "crinons" are in this case the ideals of the group algebra. In the case of non-Abelian groups, the results, while still interesting, are somewhat less far reaching than those of the usual theory. The properties and multiplication laws of the "crinons" are in this case less simple than those of the units of ideals.

E. P. Wigner (Princeton, N. J.).

Opechowski, W. Sur les groupes cristallographiques "doublès." Physica 7, 552–562 (1940). [MF 4363]

The author emphasizes the fact that, for the classification of spectral terms, etc. (in general, for the classification of solutions of problems in which the principle of superposition is valid), one has to use not the representations of the symmetry group of the problem but the representations of the group of mathematical operators which correspond to the permissible symmetry operations. This is important because, for instance, in the case of an odd number of electrons, two mathematical operators (differing in sign) correspond to the same physical operation. In consequence hereof, the group the representations of which are to be considered contains twice more elements than the symmetry group of the problem. The ideas are illustrated on several crystallographic point groups.

E. P. Wigner.

Miller, G. A. Enumeration of finite groups. Math. Student 8, 109–111 (1940). [MF 4312]

Among the classes of finite groups which have been enumerated are those having (1) small degrees (permutation groups), (2) small orders and (3) a small number of subgroups. The author discusses briefly the history of these enumerations.

J. S. Frame (Providence, R. I.).

Miller, G. A. Groups containing maximal subgroups of prime order. Proc. Nat. Acad. Sci. U. S. A. 27, 342–345 (1941). [MF 4850]

An Abelian group which contains a maximal subgroup of prime order p must be of order p^2 or pq (q prime). A non-Abelian group G containing a maximal subgroup of order p is dihedral if $p=2$, and it contains an invariant Abelian maximal subgroup of order $1+3k$ which is cyclic or of type 1^m if $p=3$. In any case G must contain an invariant subgroup of order $1+kp$, and cannot be simple. If this subgroup is non-Abelian, it must coincide with its commutator group, and hence cannot be solvable. In groups whose orders contain two or three prime factors, respectively, the proper subgroups are both maximal and minimal, or either maximal or minimal. But groups whose orders contain more than three prime factors always contain proper subgroups which are neither maximal nor minimal.

J. S. Frame.

Miller, G. A. Maximal subgroups whose orders are divisible by two or three. Proc. Nat. Acad. Sci. U. S. A. 27, 399–402 (1941). [MF 5073]

The subgroups of a complete set of n conjugate non-invariant maximal subgroups of a group G are transformed into one another according to a primitive group of degree n. This is of class $n-1$ if and only if each two of these subgroups have only identity in common. It is proved that, if the order of these subgroups is divisible by 2 or 3, then G has an Abelian invariant subgroup whose order is prime to the order of these subgroups. Next a study is made of the cross-cut of two maximal subgroups of a permutation group G of degree n, each of which is composed of the permutations leaving one letter fixed. It is shown that this cannot be invariant in the one subgroup but not in the other if G is multiply transitive, but may be if G is simply transitive. An example of the latter situation is given.

J. S. Frame.

Hua, Loo-keng and Tuan, Hsio-fu. Determination of the groups of odd-prime-power order p^n which contain a cyclic subgroup of index p^2. Sci. Rep. Nat. Tsing Hua Univ. (A) 4, 145–154 (1940). [MF 5324]

The p-groups containing a cyclic subgroup of index p^2 (p an odd prime) have been discussed by G. A. Miller [Trans. Amer. Math. Soc. 2, 259–272 (1901); 3, 383–387, 499–500 (1902)] using methods from the theory of permutations. The authors give a direct approach, representing these groups by normal forms of their generators and defining relations, and distinguishing between them by means of simple invariant properties.

R. Baer.

Lewis, F. A. A note on the special linear homogeneous group $SLH(2, p^n)$. Bull. Amer. Math. Soc. 47, 629–632 (1941). [MF 5058]

The author gives a new set of generating relations for the abstract group L simply isomorphic with $SLH(2, p^n)$. He keeps the three relations of E. H. Moore: (a) $S_\lambda S_\mu = S_{\lambda+\mu}$ (λ, μ any marks of $GF[p_n]$); (b) $T^4=I$, $S_\lambda T^2 = T^2 S_\lambda$; and (d) $(S_1T^3)^3=I$; but replaces the other relation by the simpler one (e): $(TS_\alpha TS_{2\alpha-1})^3=T^2$ for any mark $\alpha\neq 0$. A slight modification is necessary if $p>2$. Several identities are proved for these generators.

J. S. Frame.

Lubelski, S. Zur Verschärfung des Jordan-Hölderschen Satzes. Rec. Math. [Mat. Sbornik] N. S. 9 (51), 277–280 (1941). (German. Russian summary) [MF 4553]

The supplement to the Jordan-Hölder theorem given by the author is based upon the following concepts: Let G be a group, M some subgroup and N a normal subgroup of G. Then a subgroup G_1 of G shall be called a real factor with

respect to M if G_1 is normal in M and $G=N \cup G_1$, $N \cap G_1=E$. If no such G_1 exists, the quotient group G/N shall be called an ideal factor of G with respect to M. According to the Jordan-Hölder theorem any two factorizations of a finite group into simple factors have the same number of factors which are isomorphic in pairs. The author shows that one can assume that these pairs at the same time are real or ideal factors with respect to G. *O. Ore.*

Shoda, Kenjiro. Über die Invarianten endlicher Gruppen linearer Substitutionen im Körper der Charakteristik p. Jap. J. Math. **17**, 109–115 (1940). [MF 4607]

Let G be a finite group of linear transformations whose coefficients belong to a field F of prime characteristic. As was first shown by E. Noether [Nachr. Ges. Wiss. Göttingen **1926**, 28–35] there exists a finite system of integral rational invariants $J_1, J_2, \cdots, J_r$ of G such that every rational integral invariant can be expressed as a polynomial of the J_i. Shoda gives a new proof for this theorem. He also considers finite groups of linear fractional transformations (collineations) with coefficients belonging to the modular field F. In this case, there exists a finite system of rational invariants $J_1, J_2, \cdots, J_r$ such that every invariant is a rational function of the J_i. It is possible to choose the J_i in such a manner that their denominators do not contain any factors except the denominators of the transformations of G.

R. Brauer (Toronto, Ont.).

Tovbin, A. V. Sur l'existence du centre des groupes infinis et finis. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 198 (1941). [MF 4826]

The author proves the following theorems: (1) If a group B has a normal divisor B_1 of finite index with a non-trivial center which contains at least one element of prime order p, the group B has an Abelian normal subgroup of finite order p^a. (2) If a normal divisor B_1 of a group B, of finite index p^a, contains in its center an element of order p, then B contains in its center an element of order p. (3) A p-group of finite class has a non-trivial center. *J. S. Frame.*

Levi, F. W. On a method of finite combinatorics which applies to the theory of infinite groups. Bull. Calcutta Math. Soc. **32**, 65–68 (1940). [MF 5044]

The theorem that every element in a free product of any number of groups admits of a unique representation as a "reduced word" is proven as a straightforward application of elementary results on finite ordered sets. *R. Baer.*

Shü, Shien-siu. On the common representative system of residue classes of infinite groups. J. London Math. Soc. **16**, 101–104 (1941). [MF 5129]

There exists a common system of representatives for the right and for the left cosets of the group G modulo its subgroup H if, and only if, $[H:(H \cap gHg^{-1})]=[H:(g^{-1}Hg \cap H)]$ for every g in G. There exist groups G and subgroups H of G which do not satisfy this condition; it is satisfied, however, if a positive power of every element in G belongs to the normalizer of H in G; and this theorem contains the classical one, in which it is assumed that $[G:H]$ is finite, as a special case. *R. Baer* (Urbana, Ill.).

Fouxe-Rabinovitch, D. I. Über die Automorphismengruppen der freien Produkte. II. Rec. Math. [Mat. Sbornik] N. S. **9** (**51**), 183–220 (1941). (Russian. German summary) [MF 4495]

Wir wollen die Erzeugenden und die Relationen der Automorphismengruppe Γ eines freien Produktes G von n freiunzerlegbaren Faktoren A_i, die samt ihren Automorphismengruppen gegeben sind, finden. Hat G ausser $A_1, A_2, \cdots, A_m$ keine unendliche zyklische Komponenten, so lässt sich G als freies Produkt der Gruppe $H=A_{m+1}*\cdots*A_n$ und der freien Gruppe $F=A_1*A_2*\cdots*A_m$ darstellen. Sei s die grösste Anzahl der unter $A_{m+1}, \cdots, A_n$ paarweise nicht isomorphen Gruppen. Die Automorphismengruppe der Gruppe H ist in meiner Arbeit in Rec. Math. N.S. **8** (**50**), 265–276 (1940) [cf. these Rev. **2**, 215] und die der Gruppe F in den wohlbekannten Arbeiten von Nielsen [Math. Ann. **78** (1918); **79** (1919); **91** (1924)] untersucht worden. Wir beweisen, dass die Gruppe Γ durch diese zwei Gruppen, durch eine zur H isomorphe und durch s unendliche zyklische Gruppen erzeugt wird. Das vollständige Relationensystem ist im Text [§3] angeführt, und der Vollständigkeitsbeweis mit Hilfe der Nielsenschen Methode geführt.

Author's summary.

Jennings, S. A. The structure of the group ring of a p-group over a modular field. Trans. Amer. Math. Soc. **50**, 175–185 (1941). [MF 4874]

Let $\mathfrak{G}$ be a p-group, and let Γ be the group ring of $\mathfrak{G}$ with regard to a field F of characteristic p. The radical $\mathfrak{N}$ has as its basis the elements $G-1$ (G in $\mathfrak{G}$, $G\neq 1$). The elements G of $\mathfrak{G}$ for which $G-1$ belongs to $\mathfrak{N}^i$ form a characteristic subgroup $\mathfrak{R}_i$ of $\mathfrak{G}$ ($i=1, 2, \cdots$), and we have $\mathfrak{G}=\mathfrak{R}_1 \supseteqq \mathfrak{R}_2 \supseteqq \cdots \supseteqq \mathfrak{R}_{L+1}=1$, where L is the largest exponent for which $\mathfrak{N}^L\neq 0$. These groups $\mathfrak{R}_i$ are analogous to the dimension groups of Magnus [Math. Ann. **111**, 259–280 (1935); cf. H. Zassenhaus, Abh. Math. Sem. Hansischen Univ. **13**, 200–206 (1939)]. The commutator of an element of $\mathfrak{R}_i$ and of an element of $\mathfrak{R}_j$ belongs to $\mathfrak{R}_{i+j}$; the pth power of an element of $\mathfrak{R}_i$ belongs to $\mathfrak{R}_{ip}$. Conversely, the following recursive definition of $\mathfrak{R}_i$ is given: The group $\mathfrak{R}_i$ is the group generated by $(\mathfrak{R}_{i-1}, \mathfrak{G})$ and the pth powers of the elements of $\mathfrak{R}_j$, where j is the least integer which is not smaller than i/p ($i=2, 3, \cdots$; $\mathfrak{R}_1=\mathfrak{G}$). Using these subgroups $\mathfrak{R}_i$, a basis of $\mathfrak{N}^i$ is formed. If $\mathfrak{R}_i/\mathfrak{R}_{i+1}$ has the order p^{d_i}, then the rank of $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ is the coefficient of x^i in the expansion of

$$\prod_\lambda (1+x^\lambda+x^{2\lambda}+\cdots+x^{(p-1)\lambda})^{d_\lambda},$$

where λ ranges over 1, 2, 3, $\cdots$. The exponent L of the radical equals $(p-1)\sum \lambda d_\lambda$. *R. Brauer* (Toronto, Ont.).

Brauer, Richard. On sets of matrices with coefficients in a division ring. Trans. Amer. Math. Soc. **49**, 502–548 (1941). [MF 4342]

Sections 1 and 2 deal with group theory, the first with the Jordan-Hölder theorem. The connection between two composition series is studied, and it is proved that there are sets of residue systems which can be used in either composition series. The upper and lower Loewy series of a group are studied, and it is shown that the ith factor groups in both have a common constituent. This implies the theorem of Krull and Ore that both series have the same length. Section 3 is devoted to matrices with elements in a division ring (noncommutative field) K. Sections 4, 5 and 6 constitute a study of the irreducible and the Loewy constituents of a set of matrices by group-theoretic methods. In section 7 the group pairs of Pontrjagin are studied and their associated sets of matrices are defined. Section 8 deals with the regular representation $\mathfrak{R}$. If $\mathfrak{A}$ is a system of matrices forming a semigroup, $\mathfrak{A}$ and $\mathfrak{R}$ have the same irreducible constituents (except perhaps 0), and the number of Loewy constituents

in both is either the same or differs by one. A number of further results concerning the distribution of the irreducible parts of the Loewy constituents of $\mathfrak{R}$ are proved. It follows that the (left) rank r of an irreducible semigroup $\mathfrak{A}$ is divisible by the degree n, and r/n can be expressed by means of properties of the commuting set. This furnishes the basis for a proof of Wedderburn's theorem on the representation of an irreducible algebra of matrices by a total matric algebra over a division algebra. A generalization of Burnside's theorem is also obtained. In section 10 representations of sets of matrices as direct sums of subsets are studied, and in sections 11 and 12 rings $\mathfrak{A}$ of matrices of degree a are considered which contain all the scalar multiples of the unit matrix. The complete structure theory of algebras is obtained. It is proved that, if $\mathfrak{B}$ is a representation of degree b of $\mathfrak{A}$, then $\mathfrak{B}$ is a constituent of $ab\times\mathfrak{A}$. A connection is noted between the Loewy decomposition of the regular representation and the structure of the powers of the radical. *C. C. MacDuffee* (New York, N. Y.).

Cartan, Elie. Sur les groupes linéaires quaternioniens. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 191–203 (1940). [MF 4416]

A group Γ of linear transformations on n homogeneous quaternion variables, with quaternion coefficients, represents a projective group on the points of an $n-1$ dimensional quaternion projective space. Two sets of variables, X^α and $X^\alpha H$, where H is a fixed quaternion not zero, define the same point, and two transformations $A_\beta{}^\alpha$ and $B_\beta{}^\alpha$ are said to be equivalent if the one results from the other by a change of basis elements, that is: $B_\lambda{}^\alpha C_\beta{}^\lambda = C_\lambda{}^\alpha A_\beta{}^\lambda$. Each quaternion variable X^α may be expressed in terms of four real variables x^α, y^α, z^α, t^α, or two complex variables $u^\alpha = x^\alpha + iy^\alpha$, $v^\alpha = z^\alpha - it^\alpha$, in the form $X^\alpha = u^\alpha + jv^\alpha = x^\alpha + iy^\alpha + jz^\alpha + kt^\alpha$. The group Γ can thus be written as a linear group G on $4n$ real variables, which decomposes in the domain of complex numbers into two equivalent linear groups G_1 on u^α, v^α and G_2 on $-\bar{v}^\alpha$, $\bar{u}^\alpha$, which admit a skew-involution of the second kind $[u^\alpha \to -\bar{v}^\alpha, v^\alpha \to \bar{u}^\alpha]$ in a complex $2n-1$-dimensional projective space. Two types of groups Γ are distinguished: those equivalent to a group with complex (instead of quaternion) coefficients correspond to a group G which is reducible, and are said to be of second class; the others are of first class. The author proves that an irreducible group Γ of the first class corresponds to a group G which is irreducible in the real domain, but which splits in the complex domain into two equivalent groups G_1 and G_2 each irreducible in the complex domain. The group G can be associated geometrically (except for a change of basis) in only one way with a group of homographies in a projective quaternion space. Furthermore any such group G can be associated with a quaternion group Γ. *J. S. Frame.*

Janet, Maurice. Sur les formules fondamentales de la théorie des groupes finis continus. C. R. Acad. Sci. Paris **212**, 424–425 (1941). [MF 4914]

Let the composition of parameters of a Lie transformation group G be given by the functions $c_k = \phi_k(a, b)$. The differential equations of the first and second parameter groups of G, and various other differential relations, fall out of a formula which expresses the dc_k linearly in terms of certain differential forms $\omega_h(a, da)$ and $\bar{\omega}_h(b, db)$, the coefficients being functions of the c's alone. *P. A. Smith.*

Powsner, A. Über nilpotente Lie-Gruppen. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 135–142 (1940). (Russian) [MF 4741]

The basis of a nilpotent Lie group $\mathfrak{G}_r$ may be so chosen that $(Y_i, Y_k) = c_{ik}^s Y_s$, $s<k<i$. The author considers the composition functions of the parameter group and proves by induction the theorem of Cartan that these functions are polynomials in the parameters. He also gives a canonical form for the operators based on the corresponding form of a complete set of vectors and indicates a proof of G. Birkhoff's theorem that a nilpotent Lie group is representable by nilpotent matrices. *M. S. Knebelman* (Pullman, Wash.).

Hopf, Heinz. Über den Rang geschlossener Liescher Gruppen. Comment. Math. Helv. **13**, 119–143 (1940).

This paper studies the mapping $(x \to x^k)$ of a Lie group into itself (k an integer) and consists of two independent sections. The first one (§ 1) supplements an earlier paper of the author [Ann. of Math. (2) **42**, 22–52 (1941); these Rev. **3**, 61] and treats the more general case of a compact manifold Γ, where a continuous mapping $f(p, q)$ of $\Gamma\times\Gamma$ into Γ has been given. The existence of a "unit-element" e, such that $e = f(e, p) = f(p, e)$, being here assumed, put $p_0(x) = e$, $p_k(x) = f[x, p_{k-1}(x)]$; it follows from the previous paper that the homology-ring of Γ is isomorphic to that of a product $S_1 \times S_2 \times \cdots \times S_l$ of spheres S_h of odd dimension, and that to every S_h in this product there corresponds a "minimal" element V_h in the homology-ring of Γ, the V's being a basis for the minimal elements in this ring. It is shown: (a) that, for any minimal element V, $p_k(V) = k\cdot V$; (b) that, if $f(V_h) = \sum \gamma_{hi} V_i$ is the linear substitution induced on the V's by a mapping $f(x)$ of Γ into itself, the topological degree of f is the determinant $|\gamma_{hi}|$. It follows that the degree of $(x \to p_k(x))$ is k^l. The proof holds even for $k<0$ if Γ is a group.

The second part is independent of the homology theory, and studies the mapping $(x \to x^k)$ on a compact connected Lie group G by a method which presupposes a minimum of topological knowledge [the method is closely related to that used in a note by the reviewer [C. R. Acad. Sci. Paris **200**, 518–520 (1935)], which the author mentions as having been brought to his notice after completion of his paper; several results are common to both papers]. It is shown that, if A is the matrix of the adjoint representation which corresponds to $a \epsilon G$, the Jacobian of $(x \to x^k)$ at $x = a$ is $1 + A + A^2 + \cdots + A^{k-1}$, and that the determinant of this is everywhere not less than 0 (but not everywhere equal to 0, as it is not equal to 0 in the neighborhood of e). From this, using the analyticity and the constancy of the topological degree, it is deduced that $x^k = q$ has at least one solution for every $q \epsilon G$; further (by a rather involved argument), that every q in G lies on a torus-group (compact connected Abelian subgroup of G); and finally that, if λ is the dimension of a maximal torus-group, the topological degree of $(x \to x^k)$ is k^λ (which, in view of the results of § 1, shows that λ is the same as the number there denoted by l). All maximal torus-groups T_λ thus have the same dimension λ. Moreover (§ 3), every λ-dimensional compact Abelian subgroup of G must be connected, and so is a T_λ. The connected component of e in the normalizer of a now being denoted by N_a', it is found that this is the union of all T_λ's containing a; the linear manifold of tangent vectors to N_a' at e is the set of all vectors which are invariant by matrix A; the dimension of N_a' is $\equiv \lambda \pmod 2$, and so is either equal to λ (then N_a' reduces to a T_λ, 1 is a characteristic

root of A with the multiplicity λ, and a is called a regular element) or not less than $\lambda+2$. Any element, the powers of which are dense in a T_λ, is regular. If a is regular, equation $x^k=a$ has exactly k^λ solutions; if not, $x^k=a$ must have infinitely many solutions (including a manifold of dimension not less than 2), except at most for a finite number of values of k. *A. Weil* (Princeton, N. J.).

Chevalley, Claude. On the topological structure of solvable groups. Ann. of Math. (2) **42**, 668–675 (1941). [MF 4966]

If G is a connected, simply-connected, solvable Lie group and D is a discrete subgroup of its center, it is proved that G factors into the product space of a vector subgroup containing D and a Cartesian subspace. Since a connected solvable Lie group G' can be written in the form G/D, it is deduced that there is a toral subgroup T of G' and a Cartesian subspace E such that G' is homeomorphic with the product space $T\times E$ under the correspondence sending the pair (t, e) into $t\cdot e$ $(t\epsilon T,\ e\epsilon E)$. *N. E. Steenrod.*

Segal, I. E. The group ring of a locally compact group. I. Proc. Nat. Acad. Sci. U. S. A. **27**, 348–352 (1941). [4852]

If a group G has a right invariant Haar measure, then the space of all complex valued functions which belong to the Lebesgue class L_1 form a (non-commutative) ring $L(G)$ if the product of f and g is defined by the convolution

$$h(x)=\int_G f(xy^{-1})g(y)d\mu(S_y).$$

The group ring $R(G)$ is by definition the smallest ring extension of $L(G)$ which has a principal unit. The author announces highly interesting results in case G is compact or Abelian, some of which will probably hold for more general groups. For instance, the intersection of all maximal (two-sided) ideals of $R(G)$ consists only of the zero element; also every proper ideal is in a well-defined sense approximately equal to the intersection of the maximal ideals containing it, and if G is compact and the ideal is closed (in the norm of the ring) it is precisely equal to that intersection.
S. Bochner (Princeton, N. J.).

Markoff, A. On free topological groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 299–301 (1941). [MF 4833]

In this abstract there are announced, among other impressive results, the following. Every completely regular space R (in the sense, now familiar, of Tychonoff) determines uniquely a topological group F (the free topological group of the space R) with these properties: (F_1) R is a closed subspace of F; $(F_2)R$ constitutes an algebraically free system of generators of the group F; (F_3) given a continuous mapping φ of R into a topological group G, there can be found a continuous homomorphism Φ of F into G such that $\Phi x=\varphi x$ for every point x of R.

If there is defined in a completely regular space R, in any fashion, a set of abstract relations, analagous to the defining relations of a group, then there exists a uniquely determined group G realizing these relations, and a continuous mapping φ of R into G such that, if there is further given another topological group H and a continuous mapping ψ of R into H, then there exists a continuous homomorphism χ of G onto H for which $\psi=\chi\varphi$.

A third body of results concerns the unique linear topological locally convex space engendered by a completely regular R, with properties corresponding to those above of F. The note contains a short sketch of the leading ideas of the proof. *L. Zippin* (Flushing, N. Y.).

Gelfand, I. Zur Theorie der Charaktere der Abelschen topologischen Gruppen. Rec. Math. [Mat. Sbornik] N. S. **9**(51), 49–50 (1941). (German. Russian summary) [MF 4489]

If a commutative group G is group-isomorphic with a bounded set in a normed commutative ring R (the group operation in G corresponding to multiplication in R), then, for any maximal ideal M of R, the numerical function $x(M)$ is a continuous character on G, and, with the aid of an interesting lemma on generalized nilpotent elements, the author shows that, owing to the boundedness of G in R, corresponding to any two elements x_1, x_2 of G there exists an M such that $x_1(M)\neq x_2(M)$. *S. Bochner.*

Fedossejeff, M. Über einen Typus von Systemen mit zwei Operationen. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **18**, 39–55 (1940). (Russian. German summary) [MF 4760]

The two operations $*$ and $\cdot$ (product) are subject to the following postulates. (I) $*$ is one valued, commutative and associative (the unit element for $*$ is not assumed to exist). (II) Every element is idempotent with respect to $*$. (III, IV, V) The elements form an Abelian group under $\cdot$. (VI) $(a*b)\cdot c=a\cdot c*b\cdot c$. An element is called integral if $a*\epsilon=\epsilon$, and b is said to be basic if $b*\epsilon=b$ (ϵ=identity of Abelian group). Every system contains an infinite number of integral and of basic elements. Also, a is divisible by b if $a\cdot b^{-1}=c$ an integral element. If $a*b=\epsilon$, a and b are said to be relatively prime, and an element p, if such exist, is said to be prime if, for each integral element a, $p*a=\epsilon$ or $p*a=p$. Not every system has primes, but, if each integral element has only a finite number of integral factors, then the system contains primes and the law of unique decomposition into prime factors holds. The system of all integers where $\cdot$ stands for ordinary sum and $a*b=\min(a, b)$ is one of a number of realizations given in this paper.
M. S. Knebelman (Pullman, Wash.).

Gardaschnikoff, M. Über einen Typus endlicher Gruppen ohne das Assoziativgesetz. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 29–33 (1940). (Russian. German summary) [MF 4746]

The author considers a system of elements $\mathfrak{G}$ and an operation $*$ for which: (1) the result of $*$ always exists and is one valued; (2) if X, A, B are any three elements of $\mathfrak{G}$, $(X*A)*B=X*C$, where C depends only on A and B: $C=A\circ B$. Such a system is a generalized group $\mathfrak{G}(*)$ and it follows from this that the associative law for $\circ$ holds. If $\mathfrak{G}(\circ)$ is an ordinary group, then $\mathfrak{G}(*)$ has a right identity for every element and every element has a unique left inverse. If $F_1, \cdots, F_r$ are all the right identities for $\mathfrak{G}(*)$, then $\mathfrak{G}=\mathfrak{G}*F_1+\mathfrak{G}*F_2+\cdots+\mathfrak{G}*F_r$, the complex $\mathfrak{G}*F_k$ containing all elements for which F_k is a right identity. The equation $X*A=B$ will have a solution if, and only if, the right identity for A with respect to $\circ$ is the same as the right identity for B with respect to $*$. The final result obtained is that system $\mathfrak{G}$ may be obtained from an associative right group by permutating the elements in the main diagonal of the Cayley table for the group.
M. S. Knebelman (Pullman, Wash.).

Woidislawski, M. Ein konkreter Fall einiger Typen der verallgemeinerten Gruppen. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 17, 127–144 (1940). (Russian. German summary) [MF 4755]

The paper gives a number of concrete examples of some types of generalized groups. One considers complexes of functions $\varphi(x)$ in the interval (a, b) which are one-valued, monotonic increasing, $\varphi(a)=a$, $\varphi(b)=b$, so that the range of $\varphi(x)$ is (a, b). The composition rule is $\varphi_1 * \varphi_2 = \varphi_2(\varphi_1(x)) = \varphi_3$. Three types of complexes are considered: complex A consists of all continuous φ's. Complex B consists of all φ's having discontinuities such that, if x_0 is a point of discontinuity, $\varphi(x_0)=\varphi(x_0-0)$ while $\varphi(x_0+0)$ is not a value of $\varphi(x)$. Complex C consists of all φ's defined on (a, b) with the exception of a finite number of subintervals (x_1, x_2) such that $\varphi(x_1)=\varphi(x_1-0)=\varphi(x_2+0)$. It is proved that the complex A defines an ordinary group; complex B defines an associative left semi-group, while complex C defines an associative right semi-group. The different groups and semi-groups are illustrated by means of 10 figures.

M. S. Knebelman (Pullman, Wash.).

Suschkewitsch, A. Über einen Typus der verallgemeinerten Semigruppen. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 17, 19–28 (1940). (Russian. German summary) [MF 4745]

A generalized left semi-group $\mathfrak{C}$ is defined by the following postulates: (I) the product always exists and is unique; (II) the associative law holds; (III) $BA=CA$ implies $B=C$; (IV) there exists an element E such that, for each A, $AE=EA=A$; (V) for any two elements of $\mathfrak{C}$ at least one of the equations $AX=B$ or $BY=A$ is solvable. If both of these equations are solvable for a pair of elements, then A and B are of the same rank, $A\sim B$. If only the first, then $B>A$, B is of higher rank, and, if only the second, then $B<A$, B is of lower rank. The relation $A\sim B$ is symmetric, transitive and reflexive. All elements equivalent to E are called regular, are of lowest rank and the totality of regular elements forms an ordinary group $\mathfrak{G}$. The additional postulates on $\mathfrak{C}$ are: (VI) if $AB=CD$ and $A\sim C$, then $B\sim D$; (VII) the Archimedean axiom: if $B<C$ there exists an integer m such that $B^m>C$; (VIII) the order of elements of decreasing rank is finite. It is then shown that every such left semi-group $\mathfrak{C}$ can be decomposed into complexes $\mathfrak{C}=\mathfrak{G}+A\mathfrak{G}+A^2\mathfrak{G}+\cdots$, where A is not a regular element. All elements P such that, for a given A, $AP=A$ are regular and form a group $\mathfrak{G}_A\subset\mathfrak{G}$. If $A\sim B$ and $AR=B$, then R is in $\mathfrak{G}$ and $\mathfrak{G}_B=R^{-1}\mathfrak{G}_AR$. If $PA=AP_1$ and the regular elements P form any subgroup $\mathfrak{P}$ of $\mathfrak{G}$, then the P_1 form a subgroup $\mathfrak{P}_1$ and $P_1^{-1}\mathfrak{G}_AP_1=\mathfrak{G}_A$, that is, $\mathfrak{G}_A$ is an invariant subgroup of $\mathfrak{P}_1$ and $\mathfrak{P}$ is simply isomorphic with the factor group $\mathfrak{P}_1/\mathfrak{G}_A$. The paper concludes by showing that the set of "deficient" substitutions on a denumerable number of letters is such a semi-group (a deficient substitution is one in which a finite number of letters may be missing). The ordinary substitutions form the group $\mathfrak{G}$ of regular elements. The set of all orthogonal and left semi-orthogonal matrices also form such a semi-group.

M. S. Knebelman (Pullman, Wash.).

Krasner, Marc. La caractérisation des hypergroupes de classes et le problème de Schreier dans ces hypergroupes. C. R. Acad. Sci. Paris 212, 948–950 (1941). [MF 5038]

A hypergroup H is said to be a hypergroup of classes, or a hypergroup$_D$ [see, for example, the author's earlier papers, Duke Math. J. 6, 120–140 (1940); 7, 121–135 (1940); these Rev. 1, 260; 2, 123], if H is isomorphic to the hypergroup of right cosets of a subgroup g in a group G, and then the pair (G, g) is called a representation of H, an irreducible representation if g has no subgroup, other than identity, which is invariant in G. The author here states, without proofs, necessary and sufficient conditions that H be a hypergroup$_D$, provided H contains a bilateral unit e, scalar on the right, that is, for every $c\epsilon H$, $ce=c$ [misprinted as $ce=e$], $c\epsilon ec$. The existence of irreducible representations of H is said to follow from the same conditions, and there is a unique irreducible representation if H is finite. In terms of these irreducible representations of two hypergroups$_D$ h and H, the author states a solution of the problem of Schreier: to construct, apart from isomorphism, for two given hypergroups$_D$ h and H all hypergroups$_D$ $\mathcal{K}$ which contain h as a sub-hypergroup and are such that $\mathcal{K}/h=H$. *R. Hull.*

ANALYSIS

Fourier Series and Integrals, Integral Transforms

Quade, W. Abschätzungen zur trigonometrischen Interpolation. Deutsche Math. 5, 482–512 (1941). [MF 4800]

The author considers the trigonometric interpolation polynomials of a continuous periodic function $f(x)$ taken at the equidistant abscissas. He gives numerous estimations between the difference of the function and its interpolation polynomials. *P. Erdös* (Philadelphia, Pa.).

Denjoy, Arnaud. La convergence en moyenne absolue des séries trigonométriques. Bull. Sci. Math. (2) 64, 147–153 (1940). [MF 5230]

The paper contains a new proof of the strong summability of Fourier series of functions of the class L^2. As an application of the method used, the following theorem is proved: for any increasing sequence of numbers $0, \alpha_1, \alpha_2, \cdots, \alpha_r, 1$, the integral

$$\int_0^\infty |1-2\cos\alpha_1\theta+2\cos\alpha_2\theta-\cdots +(-1)^r 2\cos\alpha_r\theta+(-1)^{r+1}\cos\theta|\,d\theta/\theta^2$$

is bounded by a constant independent of r and of the α_i's.

R. Salem (Cambridge, Mass.).

Cameron, Robert H. Some introductory exercises in the manipulation of Fourier transforms. Nat. Math. Mag. 15, 331–356 (1941). [MF 5242]

Expository article.

von Neumann, J. and Schoenberg, I. J. Fourier integrals and metric geometry. Trans. Amer. Math. Soc. 50, 226–251 (1941). [MF 5142]

If f_x is an isometric map of the straight line $-\infty<x<\infty$ into Hilbert space, then the distance in Hilbert space

between two points f_x and f_y is a function $F(t)$ depending only on the quantity $t=|x-y|$. The paper gives two proofs, one operational and one analytical, for the formula

$$F^2(t)=\int_0^\infty ((\sin tu)^2/u^2)d\gamma(u),$$

where $\gamma(u)$ is a monotone function in $0\leqq u<\infty$ and $\int_1^\infty d\gamma(u)/u^2$ converges. If we replace the straight line by a Euclidean space of any finite dimension m, in which case the quantity t is the Euclidean distance $(\sum_{k=1}^m(x_k-y_k)^2)^{\frac{1}{2}}$, then the formula generalizes to

$$F^2(t)=\int_0^\infty ((1-\Omega_m(tu))/u^2)d\gamma(u),$$

where $\Omega_m(t)$ is a Bessel function. The generalization is proved only by the second method. Finally it should be noted that, according to a previous result of Schoenberg, the limit $m\to\infty$ in the latter formula leads to $\int_0^\infty((1-e^{-t^2u})/u)d\gamma(u)$.

S. Bochner (Princeton, N. J.).

Kober, H. On Dirichlet's singular integral and Fourier transforms. Quart. J. Math., Oxford Ser. **12**, 78–85 (1941). [MF 5219]

The author raises the following questions. If

$$f(t)\epsilon L_1(-\infty,\infty),$$

(a) under what conditions does the Dirichlet transform

$$D_\alpha f=\frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin\alpha(s-t)}{s-t}f(t)dt$$

belong to L_1; (b) under what conditions is the Fourier transformation, applied to $f(t)$, inversible in the sense of mean convergence, that is, when is

$$\lim_{\alpha\to\infty}\left\|f(t)-(2\pi)^{-\frac{1}{2}}\int_{-\alpha}^{\alpha}\phi(t)e^{itx}dt\right\|_1=0,$$

where

$$\phi(t)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty f(x)e^{-itx}dx.$$

He shows that $\phi(\alpha)=\phi(-\alpha)=0$ is a necessary but not sufficient condition that $D_\alpha f\epsilon L_1$. The non-sufficiency will be shown by H. R. Pitt in a forthcoming paper. If $f(t)\epsilon L_1(-1,1)$ and there exists a $\rho>0$ such that $|t|^\rho f(t)\epsilon L_1(-\infty,\infty)$, and $\phi(\alpha)=\phi(-\alpha)=0$, then $D_\alpha f\epsilon L_1$, and, more generally, when $0\leqq\sigma<\tau=\min(1,\rho)$ and $r(1-\sigma+\tau)>1$, then $|s|^\sigma D_\alpha f\epsilon L_r(-\infty,\infty)$. A necessary and sufficient condition for (b) to hold is that $f(t)$ be equivalent to an entire function of exponential type (finite Fourier integral). [For the corresponding problem in L_p, see Hille and Tamarkin, Bull. Amer. Math. Soc. **39**, 768–774 (1933).]

E. Hille.

Hardy, G. H. A double integral. J. London Math. Soc. **16**, 89–94 (1941). [MF 5126]

Let $K(x)$ be a Fourier kernel, that is, $K(x)$ real and integrable in every finite interval $(0,X)$, $K_1(x)=\int_0^x K(t)dt$ satisfying

$$\int_0^\infty K_1(ax)K_1(bx)x^{-2}dx=\min(a,b)$$

for all positive a and b. If $K_1(x)=O(x^{\frac{1}{2}})$ for all x, and $H(x,y)$ is homogeneous of degree -1 and has partial derivatives H_x and H_y continuous except at the origin, then

$$\int_0^\infty\int_0^\infty K(ax)K(by)H(x,y)dxdy=H(b,a),\quad a,b>0.$$

The integral is a repeated Cauchy limit. If also $K_1(x)=o(x^{\frac{1}{2}})$ at 0 and ∞, and $H(x,y)$ has second partial derivatives continuous except at the origin, the same relation holds with the integral interpreted as

$$\lim_{\xi\to0,\,X\to\infty,\,\eta\to0,\,Y\to\infty}\int_\xi^X\int_\eta^Y.$$

The condition $K_1(x)=o(x^{\frac{1}{2}})$ cannot be relaxed.

R. P. Boas, Jr. (Durham, N. C.).

Meijer, C. S. Eine neue Erweiterung der Laplace-Transformation. I. Nederl. Akad. Wetensch., Proc. **44**, 727–737 (1941). [MF 5119]

The author introduces the transformation

$$(1)\qquad f(s)=\int_0^\infty e^{-\frac{1}{2}st}W_{k+\frac{1}{2},m}(st)(st)^{-k-\frac{1}{2}}F(t)dt$$

and its inverse

$$(2)\quad F(t)=\lim_{\lambda\to\infty}\frac{\Gamma(1-k+m)}{2\pi i\Gamma(1+2m)}\int_{\beta-\lambda i}^{\beta+\lambda i}e^{\frac{1}{2}ts}M_{k-\frac{1}{2},m}(ts)(ts)^{k-\frac{1}{2}}f(s)ds,$$

where $M_{k,m}(z)$ and $W_{k,m}(z)$ are the two Whittaker functions. When $k\leqq m\leqq -k$ he obtains sufficient conditions on $F(t)$ under which (2) follows from (1). Also, sufficient conditions on $f(s)$ are given under which (1) is a consequence of (2), when $\Re(k)\leqq-\Re(m)<\frac{1}{2}$. The sets of conditions here are known conditions for the validity of the complex inversion formula of the Laplace transformation. In fact, when $k=-m$, (1) and (2) reduce to the Laplace transformation and this inversion formula, respectively. Some reciprocal relations between special functions are given as particular cases of (1) and (2).

R. V. Churchill.

Royall, Norman N., Jr. Laplace transforms of multiply monotonic functions. Duke Math. J. **8**, 546–558 (1941).

The purpose of the paper is the proof of analogues of certain theorems of Fejér and the reviewer (for Laplace transforms instead of power series). Let $(-1)^n\alpha^{(n)}(t)\geqq0$, $n=0,1,\cdots,N$, and

$$f(s,R)=\int_R^\infty e^{-st}\alpha(t)dt,\qquad f(s,0)=f(s).$$

Then (1) for $N=3$, $\Re s>0$, $|e^{sR}f(s,R)|$ is decreasing as $R\uparrow\infty$. (2) For $N=2$ the same holds for $|f(s,R)|$. (3) If $N=3$, $\alpha(t)\not\equiv0$, the function $f(s)$ is univalent for $\Re s>0$. (4) The latter statement is not true if $N=2$.

G. Szegö (Stanford University, Calif.).

Bernstein, Dorothy L. The double Laplace integral. Duke Math. J. **8**, 460–496 (1941).

The author studies functions of the form

$$(1)\qquad f(s,t)=\int_0^\infty\int_0^\infty e^{-sx-ty}d_xd_y\varphi(x,y),$$

where s and t are complex variables, $\varphi(x, y)$ is a function of bounded variation (in a suitable sense) in every finite rectangle and the integral is defined as $\lim_{S,T\to\infty} \int_0^S\int_0^T$. Some properties of (1) are analogues of known properties of single Laplace integrals; others are rather different. Much of the theory of double Dirichlet series generalizes to (1). Some of the author's results have been previously stated by other writers in more or less complete, and occasionally incorrect, form (the author gives an extensive bibliography). This paper, however, seems to be the first systematic exposition of any substantial part of the subject. As in the theory of double Dirichlet series, the useful kind of convergence for (1) is bounded convergence. The existence of associated abscissas of bounded convergence is established, and several formulas for them are obtained. Absolute convergence and uniform bounded convergence are also discussed. There is a detailed discussion of complex inversion formulas for (1). Let

$$\bar{\varphi}_{\rho_1,\rho_2}(w, z)=\frac{1}{\Gamma(\rho_1)\Gamma(\rho_2)}\int_0^z\int_0^w (w-x)^{\rho_1-1}(z-y)^{\rho_2-1}\varphi(x, y)dxdy$$

($\rho_1>0$, $\rho_2>0$) with corresponding definitions, involving right- and left-hand limits of φ, when ρ_1 or ρ_2 is zero. Let $V_\varphi'[a, b; y]$ denote the total variation with respect to x of $\varphi(x, y)$ on $a\leqq x\leqq b$, with a similar definition of $V_\varphi''[x; c, d]$. If M, a_0, b_0 are real and

$$\text{(A)} \qquad |\varphi(x, y)|<Me^{a_0x+b_0y}$$

for all non-negative x and y and if for some positive δ

$$\text{(B)} \qquad V_\varphi'[w-\delta, w+\delta; y]<Me^{b_0y}$$

and

$$\text{(C)} \qquad V_\varphi''[x; z-\delta, z+\delta]<Me^{a_0x}$$

for all non-negative y and x, respectively; then

$$\text{(2)} \qquad \bar{\varphi}_{\rho_1,\rho_2}(w, z)=\frac{-1}{4\pi^2}\int_{b-i\infty}^{b+i\infty}\int_{a-i\infty}^{a+i\infty}\frac{e^{sw+tz}f(s, t)}{s^{\rho_1+1}t^{\rho_2+1}}dsdt$$

for $\rho_1\geqq 0$, $\rho_2\geqq 0$, $a>a_0$, $b>b_0$, $a\neq 0$, $b\neq 0$; the integral is a double Cauchy principal value. Other conditions under which (2) is true are obtained; in particular, (A) is true if a_0 and b_0 are non-negative numbers such that (a_0, b_0) is within the region of bounded convergence of (1); if $\rho_1>1$ and $\rho_2>1$, (B) and (C) may be dropped. [The author points out an error in the proof by which Durañona y Vedia and Trejo claim to have proved (2) for $\rho_1\geqq 0$ and $\rho_2\geqq 0$ using only (A).] Inversion of (1) by repeated integrals is also discussed. *R. P. Boas, Jr.* (Durham, N. C.).

Crum, M. M. On the resultant of two functions. Quart. J. Math., Oxford Ser. **12**, 108–111 (1941). [MF 5222]

The author has a new proof for a theorem of Titchmarsh that for $f(x)$ and $g(x)$ in $L(0, c)$ the resultant $\int_0^x f(y)g(x-y)dy$ is null in $(0, c)$ only if $f(x)$ is null in $(0, a)$ and $g(x)$ is null in $(0, b)$, where $a+b\geqq c$. Any proof of the theorem will of necessity involve an investigation of growth of analytic functions of type $F(z)=\int_0^c f(t)e^{zt}dt$, since it amounts to stating that for any two functions of this type the indicator diagram of the product is the precise logical sum of those of the factors. The proof of the author is simpler than that of Titchmarsh insofar as it avoids Hadamard factorization of the functions involved. *S. Bochner* (Princeton, N. J.).

Tricomi, Francesco. Sul "principio del ciclo chiuso" del Volterra. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 74–82 (1941). [MF 4793]

Consider the transformations

$$V_x[F(y)]=F(x)-\lambda\int_{-\infty}^{x} K(x, y)F(y)dy$$

and

$$W_x[F(y)]=-\lambda\int_{-\infty}^{x} K(x, y)F(y)dy.$$

Suppose that $K(x, y)$ is continuous and that for any x, $T>0$ the series $\sum_{n=1}^{\infty} K(x, s-nT)$ converges uniformly in $x\leqq s\leqq x+T$. The author shows that the above transformations will transform an arbitrary periodic $F(y)$ into another periodic function with the same period if, and only if, $K(x, y)=N(x-y)$. Another necessary and sufficient condition is that $V_x[F(y-h)]=V_{x-h}[F(y)]$. *W. Feller.*

Erdélyi, A. On the connection between Hankel transforms of different order. J. London Math. Soc. **16**, 113–117 (1941). [MF 5132]

Let all functions belong to $L^2(0, \infty)$. $\mathfrak{H}_\nu f(x)$ denotes the "cut" Hankel transform of $f(x)$ of order ν [cf. Kober and Erdélyi, Quart. J. Math., Oxford Ser. 11, 212–221 (1940); these Rev. **2**, 192]. R_ν denotes the class of pairs of functions f and F of L^2 such that $F=\mathfrak{H}_\nu f$; $\mathfrak{M}f(x)$ denotes the Mellin transform of $f(x)$. The author discusses operations carrying R_μ into R_ν, thus generalizing the notion of self-reciprocal functions. He gives two general rules: let $F=\mathfrak{H}_\mu f$, $\mathfrak{K}=\mathfrak{M}K$, $\mathfrak{k}=\mathfrak{M}k$; then

$$\int_0^\infty K(xy)F(y)dy=H_\nu\int_0^\infty k(xy)f(y)dy$$

if and only if

$$(*) \qquad \Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}-it)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-it)\mathfrak{K}(t) =\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}+it)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}+it)\mathfrak{k}(-t);$$

and

$$\int_0^\infty K(x/y)F(y)dy/y=H_\nu\int_0^\infty k(x/y)f(y)dy/y$$

if and only if (*) holds with the first factor on the left interchanged with the first factor on the right. Several representations for kernels satisfying these rules are given, for example, $K(x)=I^+_{\frac{1}{2}\mu,\,\frac{1}{2}\nu-\frac{1}{2}\mu}[h(x)]$, $k(x)=I^+_{\frac{1}{2}\mu,\,\frac{1}{2}\nu-\frac{1}{2}\mu}[x^{-1}h(x^{-1})]$, where $h(x)$ is arbitrary in L^2, and I^+ is one of Kober's operators of fractional integration and differentiation [Quart. J. Math., Oxford Ser. 11, 193–211 (1940); these Rev. **2**, 192]. It is also explained how the various transformations considered can be regarded as formed in two steps, one carrying R_μ into R_ν by a particular transformation, the other being a general transformation of R_ν into itself. *R. P. Boas, Jr.* (Durham, N. C.).

Shanker, Hari. A note on certain self-reciprocal functions. J. Indian Math. Soc. (N.S.) **5**, 44–45 (1941). [MF 5113]

The self-reciprocal functions rediscovered by Dhar [J. Indian Math. Soc. (2) 4, 91–96 (1940); these Rev. **2**, 192] are exhibited as special cases of an integral evaluated by Varma [J. Indian Math. Soc. (2) 3, 25–33, 54–55 (1938)]. *R. P. Boas, Jr.* (Durham, N. C.).

Kober, H. On a theorem of Schur and on fractional integrals of purely imaginary order. Trans. Amer. Math. Soc. **50**, 160–174 (1941). [MF 4873]

The author considers two types of generalized fractional

integrals [see Kober, Quart. J. Math., Oxford Ser. 11, 193–211 (1940); cf. these Rev. 2, 191]

$$f^{+}_{\eta,\alpha}(y)=I^{+}_{\eta,\alpha}f=(y^{-\eta-\alpha}/\Gamma(\alpha))\int_0^y (y-x)^{\alpha-1}x^{\eta}f(x)dx,$$

$$f^{-}_{\eta,\alpha}(y)=J^{-}_{\eta,\alpha}f=(y^{\eta}/\Gamma(\alpha))\int_y^{\infty} (x-y)^{\alpha-1}x^{-\eta-\alpha}f(x)dx,$$

which define bounded linear transformations on $L_p(0, \infty)$ to L_p for $1\leqq p\leqq \infty$ when $\Re(\alpha)>0$ and $\Re(\eta)>-1/p'$ or $-1/p$, respectively. They are meaningless for $\Re(\alpha)=0$, but the author now shows that, if $\Re(\eta)>-\frac{1}{2}$, $\alpha\rightarrow\beta$, $\Re(\beta)=0$, then $I^{+}_{\eta,\alpha}f$ and $J^{-}_{\eta,\alpha}f$ converge strongly in L_2 to limits $I^{+}_{\eta,\beta}f$ and $J^{-}_{\eta,\beta}f$. These limits are certain Mellin transforms and define bounded linear transformations on L_2 to L_2. Conversely, if $I^{+}_{\eta,\beta}f=g$, $J^{-}_{\eta,\beta}f=h$, then $f=I^{+}_{\eta+\beta,-\beta}g=J^{-}_{\eta+\beta,-\beta}h$. The classical fractional integrals of Riemann-Liouville and H. Weyl are $y^{\alpha}I^{+}_{0,\alpha}f$ and $y^{\alpha}J^{-}_{-\alpha,\alpha}f$, $\Re(\alpha)>0$, in the author's notation. They also admit of limits on $\Re(\alpha)=0$. The author defines $X_{\beta}f=x^{\beta}I^{+}_{0,\beta}f$ and $Y^{\beta}f=y^{\beta}J^{-}_{-\beta,\beta}f$ for $f\epsilon L_2$. He shows that X_{β} and Y_{β} are bounded linear transformations on L_2 to L_2 which form groups with respect to the parameter β.

The extensions to L_p are rather limited, especially for $1\leqq p<2$. We quote: If $\Re(\zeta)>-1/p'$, $1\leqq p<\infty$, $f(x)\epsilon L_p$ and $\alpha\rightarrow 0$ in a sector $|\arg\alpha|\leqq\theta<\pi/2$, then $\|I^{+}_{\zeta,\alpha}f-f\|_p\rightarrow 0$. Furthermore, $X_{\beta}f$ exists with domain and range in $\mathfrak{M}_p$, the subspace of L_p made up of functions which are Mellin transforms of functions in $L_{p'}(-\infty, \infty)$, $2\leqq p\leqq\infty$, and in this subspace X_{β} forms a group and $\|X_{\beta}-X_{\beta_0}\|_p\rightarrow 0$ as $\beta\rightarrow\beta_0$. Similarly for Y_{β} when $p<\infty$. It is shown that X_{β}, $\beta\neq 0$, with domain $\mathfrak{M}_p$, $2\leqq p\leqq\infty$, has no characteristic values, while the characteristic values of Y_{β} are the point set $\exp(-(\pi/2)|\beta|)<|\lambda|<\exp((\pi/2)|\beta|)$ in the λ-plane, each value having infinitely many characteristic functions. The proofs are partly based upon an extension of a theorem of I. Schur on transformations $Wf=\int_0^{\infty}K(x, y)f(x)dx$, where $K(x, y)$ is homogeneous of degree -1, $K(x, 1)x^{-\frac{1}{2}}\epsilon L_1$. Also Wf is bounded on L_2 and the best bound is determined.

E. Hille (Palo Alto, Calif.).

Differential Equations

Leemans, J. Sur les équations différentielles linéaires à coefficients constants. Mathesis 54, 152–155 (1940). [MF 5157]

This paper contains a brief exposition of the solution of linear differential equations with constant coefficients by operational methods.

W. T. Reid (Chicago, Ill.).

Rosenblatt, Alfred. Sur les points singuliers des équations différentielles. Revista Ci., Lima 43, 205–251 (1941). [MF 4849]

In four earlier papers [Goettingen, W. Fr. Kästner, 1908; Prace Mat.-Fiz. 20 (1908); C. R. Acad. Sci. Paris 158 (1914); Prace Mat.-Fiz. 27 (1916)] this author has studied the fixed singular points of differential equations. Since these earlier works are not all readily obtainable and since the results were presented in compact and somewhat incomplete form, the author has written the present paper to give more detailed demonstrations and results along the lines of the former works. More specifically, the present paper gives a detailed development of conditions under which the system of two ordinary first order equations

$$(\sum A y_1^{\alpha_1}y_2^{\alpha_2}x^{\beta})\frac{dy_1}{dx}=\sum A^1y_1^{\alpha_1^1}y_2^{\alpha_2^1}x^{\beta_1},$$

$$(\sum A y_1^{\alpha_1}y_2^{\alpha_2}x^{\beta})\frac{dy_2}{dx}=\sum A^2y_1^{\alpha_1^2}y_2^{\alpha_2^2}x^{\beta_2}$$

has integrals of the form

$$y_1=\nu_1x^{\rho_1}e^{-k_1(\log 1/x)^m},\quad y_2=\nu_2x^{\rho_2}e^{-k_2(\log 1/x)^m},\quad 0<m<1.$$

W. M. Whyburn (Los Angeles, Calif.).

Rosenblatt, Alfred. Sur l'unicité des solutions des équations différentielles ordinaires. Revista Ci., Lima 43, 75–93 (1941). [MF 4842]

E. J. McShane [Bull. Amer. Math. Soc. 45, 755–757 (1939)] gave uniqueness theorems for the differential system $dy_i/dx=f_i(x, y_1, \cdots, y_n)$, $i=1, \cdots, n$. Tonelli, Rosenblatt, Scorza-Dragoni, Zwirner and many others have given uniqueness theorems for systems of this type. The present paper gives a brief resumé of these theorems and then develops further such theorems. One of these states that the system $y'=f(x, y, z)$, $z'=g(x, y, z)$, $y(0)=z(0)=0$, where f and g are continuous in $0\leqq x\leqq a$, $|y|\leqq H$, $|z|\leqq H$, has a unique solution provided the following inequalities hold:

$$|f(x, y_1, z_1)-f(x, y_2, z_2)|\leqq 2|z_2-z_1|/x,$$

$$|g(x, y_1, z_1)-g(x, y_2, z_2)|\leqq k|y_1-y_2|,\quad 0<k<1.$$

Certain generalizations of this theorem are given.

W. M. Whyburn (Los Angeles, Calif.).

Zaremba, Stanislas Christian. Sur une question relative aux intégrales premières des systèmes d'équations différentielles. J. Math. Pures Appl. (9) 19, 411–426 (1940). [MF 4631]

This paper concerns certain questions raised by M. T. Wazewski [Ann. Soc. Polon. Math. 16, 145–161 (1938)] for differential systems of the type

$$dx_1/X_1=dx_2/X_2=\cdots=dx_{n+1}/X_{n+1},\ X_i=X_i(x_1, \cdots, x_{n+1}).$$

The specific question answered in the present paper has to do with the existence of a differential system of the above type which is without singularities in a unicoherent domain but which has closed characteristics in this domain and which has n independent first integrals in that domain. A second order example is constructed in such a way that it proves this existence. The author actually solves the equivalent problem in implicit function form by constructing two functions $F(x, y, z)$, $G(x, y, z)$ which are independent in a unicoherent domain and both of which are constant on a closed curve lying interior to this domain.

W. M. Whyburn (Los Angeles, Calif.).

Picone, Mauro. Nuova analisi esistenziale e quantitativa delle soluzioni dei sistemi di equazioni differenziali ordinarie. Ann. Scuola Norm. Super. Pisa (2) 10, 13–36 (1941). [MF 5470]

Let $\{y_h^{(0)}(x)\}$ be a solution of

$$(*)\quad\begin{cases}\dfrac{dy_h}{dx}=f_h(x; y_1, \cdots, y_h; \lambda_1, \cdots, \lambda_q), & h=1, \cdots, p,\\ y_h(x_0)=a_h\end{cases}$$

for a given set of parameters $\{\lambda_i^{(0)}\}$ and initial values $\{a_h^{(0)}\}$. The "theory of the solutions infinitesimally near" deals with the problem of the existence of a solution $\{y_h\}$ of (*) when the parameters and the initial values are varied, also with the magnitude of $\sum_{h=1}^{n}|y_h-y_h^{(0)}|$. The present paper treats these same problems without the hypothesis that $\{y_h^{(0)}\}$ is a solution of (*). The treatment is based on Picard's method of successive approximations. To the hypotheses of line one, page 29, add "e se $\varphi_h(x)\geqq 0$ per ogni h."

F. G. Dressel (Durham, N. C.).

Cesari, Lamberto. Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (6) 11, 633–695 (1940). [MF 4992]

This paper is concerned with a linear homogeneous differential system of the form

$$y_i'(x)=\sum_{h=1}^{n}[a_{ih}+\lambda\phi_{ih}(x)]y_h(x), \quad i=1,\cdots,n,$$

where the a_{ih} are constants, λ is a real parameter and the $\phi_{ih}(x)$ are periodic functions of the real variable x such that the integrals of these functions over a period are equal to zero. Various conditions are given which assure that there exists a neighborhood of $\lambda=0$ such that for λ in this neighborhood all the solutions of the system are stable.

W. T. Reid (Chicago, Ill.).

Cesari, Lamberto. Un nuovo criterio di stabilità per le soluzioni delle equazioni differenziali lineari. Ann. Scuola Norm. Super. Pisa (2) 9, 163–186 (1940). [MF 4994]

The author shows that all the solutions of a linear homogeneous differential system

$$y_\lambda'(x)=\sum_{\mu=1}^{n}[a_{\lambda\mu}+f_{\lambda\mu}(x)+\phi_{\lambda\mu}(x)]y_\mu(x), \quad \lambda=1,\cdots,n,$$

are stable if: (i) the functions $f_{\lambda\mu}(x)$ are of bounded variation on (x_0,∞) and $\lim_{x\to\infty}f_{\lambda\mu}(x)=0$ ($\lambda,\mu=1,\cdots,n$); (ii) the functions $\phi_{\lambda\mu}(x)$ are absolutely integrable on (x_0,∞); (iii) the constants $a_{\lambda\mu}$ are such that the real part of each root of the equation $|a_{\lambda\mu}-\delta_{\lambda\mu}\rho|=0$ is non-positive and each root whose real part is zero is a simple root; (iv) on (x_0,∞) the real part of each root of the equation

$$|a_{\lambda\mu}+f_{\lambda\mu}(x)-\delta_{\lambda\mu}\rho(x)|=0$$

is non-positive. Correspondingly, all the solutions of the nonhomogeneous system

$$y_\lambda'(x)=\sum_{\mu=1}^{n}[a_{\lambda\mu}+f_{\lambda\mu}(x)+\phi_{\lambda\mu}(x)]y_\mu(x)+\phi_\lambda(x)$$

are stable if in addition to the above hypotheses we also have: (v) $\lim_{x\to\infty}\phi_\lambda(x)=b_\lambda$ exists and the functions $\phi_\lambda(x)-b_\lambda$ are absolutely integrable on (x_0,∞); (vi) either all the roots of $|a_{\lambda\mu}-\delta_{\lambda\mu}\rho|=0$ are different from zero or $b_\lambda=0$ ($\lambda=1,\cdots,n$). From these results the author deduces corresponding criteria for the stability of the solutions of a single linear differential equation of the nth order.

W. T. Reid (Chicago, Ill.).

Trjitzinsky, W. J. Properties of growth for solutions of differential equations of dynamical type. Trans. Amer. Math. Soc. 50, 252–294 (1941). [MF 5143]

The first part of this paper deals with the differential system

(1) $\qquad dx_i/dt=p_{1i}(t)x_1+\cdots+p_{ni}(t)x_n, \quad i=1,2,\cdots,n,$

either in this scalar form or in the matrix form

(2) $\qquad X^{(1)}(t)=X(t)P(t).$

The interval is $t_0\leqq t$, and the $p_{ij}(t)$ are taken to be continuous. A real continuous monotone function $\psi(t)$ such that $\psi(t)\to\infty$ at $t\to\infty$ is called "admissible." A constant λ is called a characteristic number of a function $x(t)$ relative to an admissible $\psi(t)$ if

$$\overline{\lim_{t\to\infty}}|x\psi^{\lambda+\epsilon}|=\infty, \quad \lim_{t\to\infty}|x\psi^{\lambda-\epsilon}|=0,$$

for all $\epsilon>0$, and the least such number for functions $x_1(t),\cdots,x_n(t)$ is called the characteristic number of the set $(x_1,\cdots,x_n)$.

In part the results obtained relative to (1) are the following: If $p(t)$ is the "least" function such that $|p_{ij}(t)|\leqq p(t)$, and if $\tau=\int_{t_0}^{t}p(t)dt$, every solution of (1) has a finite characteristic number relative to e^t or e^τ according as $\lim_{t\to\infty}\tau$ is finite or infinite. If $|p_{ij}(t)|\leqq b$, the characteristic number of every solution relative to e^t lies between the values $\pm(2n-1)b$. If $p_{ij}(t)\to 0$ as $t\to\infty$, then, for all $\epsilon>0$, $\lim_{t\to\infty}ve^{\epsilon t}=\infty$, $\lim_{t\to\infty}ve^{-\epsilon t}=0$, where $v=x_1^2+\cdots+x_n^2$.

In part the results obtained relative to (2) (by product integration) are: If X_0 is the solution determined by the condition $X_0(t_0)=I$, its elements satisfy the relations

$$|x_{ij}(t)-\delta_{ij}|\leqq(-1+e^{n\tau})/n.$$

If $\tau_1=\lim_{t\to\infty}\tau$ is finite, there exists a solution such that $X(\infty)=I$, and its elements satisfy the relations

$$|x_{ij}(t)-\delta_{ij}|\leqq(-1+e^{n(\tau_1-\tau)})/n.$$

If $p_{ij}(t)\geqq\rho(t)\geqq 0$, the elements of $X_0(t)$ satisfy the relations

$$x_{ij}(t)-\delta_{ij}\geqq\int_{t_0}^{t}\rho(t)dt.$$

If $|p_{ij}(t)|\leqq w(t)e^{h_i(t)-h_j(t)}$ with $w(t)$ continuous and the $h_i(t)$ differentiable and such that $h_1^{(1)}(t)\leqq h_2^{(1)}(t)\leqq\cdots\leqq h_n^{(1)}(t)$, or $h_1^{(1)}(t)\geqq h_2^{(1)}(t)\geqq\cdots\geqq h_n^{(1)}(t)$, the elements of $X_0(t)$ satisfy the relations

$$|x_{ij}(t)-\delta_{ij}|\leqq e^{h_i(t)-h_j(t)}\left\{-1+\exp\left(n\int_{t_0}^{t}wdt\right)\right\}\Big/n.$$

In the latter part of the paper the deductions are extended to non-linear systems of the form

$$dx_i/dt=f_i(t,x_1,\cdots,x_n)=l_i+g_i,$$

in which $l_i=p_{1i}(t)x_1+\cdots+p_{ni}(t)x_n$,

$$g_i=\sum_{i_1+\cdots+i_n=2}p_i^{i_1,\cdots,i_n}(t,x_1,\cdots,x_n)x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}.$$

The hypotheses, which will not be repeated here, are such as to include the cases in which the f_i are analytic in $x_1,\cdots,x_n$ and continuous in t for $|x_i|\leqq H$, $t\geqq t_0$.

R. E. Langer (Madison, Wis.).

Panovko, J. G. On the general solution of the problem of constrained oscillations of systems with several degrees of freedom. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] 5, no. 1, 103–108 (1941). (Russian. English summary) [MF 4659]

The representation for the problem in question is taken in the form

$$y_i(t)=\sum_{k=1}^{n}\int_0^t P_k(\tau)R_{i,k}(t,\tau)d\tau.$$

The author establishes a reasonably simple way of con-

structing the $R_{i,k}$ on the basis of methods of operational calculus, leading to formation of a suitable schematic model, with the aid of which the determination of the $R_{i,k}$ is facilitated. *W. J. Trjitzinsky* (Urbana, Ill.).

Haag, J. Méthode de calcul des oscillations mécaniques ou électriques. Application aux filtres. J. Math. Pures Appl. (9) 19, 107–120 (1940). [MF 4617]

This paper is concerned with mechanical systems (S) depending on n parameters q_ν which are assumed (1) sinusoidal functions of the time: $q_\nu = Re^{-\beta t}\cos(\alpha t+\varphi)$, and where (2) the kinetic energy $2T$ is a quadratic form in the $\dot q_\nu (=dq_\nu/dt)$ with constant coefficients, (3) (S) is subjected to permanent elastic forces derivable from a potential V which is a quadratic form in the q_ν with constant coefficients, and (4) (S) is subjected to permanent viscous resistances derivable from a potential of velocities which is a quadratic form in the $\dot q_\nu$ with constant coefficients. The discussion is carried on in the complex plane by the convention: replace every real sinusoidal $x(t)=Re^{-\beta t}\cos(\alpha t+\varphi)$ by $z(t)=Re^{-\beta t}e^{i(\alpha t+\varphi)}=Re^{i(\omega t+\varphi)}$, $\omega=\alpha+i\beta$.

It is shown that the (complex) quadratic form

$$Z=2S+2i(\omega T-V/\omega),$$

called the quadratic impedance of (S), is basic for such systems. For instance, applying the above complex-convention, Z is written as $Z=\sum z^{jk}\nu_j\nu_k$, where ν_j is the complex correspondent of $\dot q_j$, and the Lagrange equations become $\partial Z/\partial\nu_j=0$. These are compatible only if the discriminant $D(\omega)$ of Z is zero, and to each of the $2n$ roots of $D(\omega)=0$ corresponds a proper oscillation of (S) which, in case S is positive definite, reduces to n damped oscillations of arbitrary phase. Under external sinusoidal forces of pulsation ω the Lagrange equations for (S) are $f^{(j)}=\frac{1}{2}\partial Z/\partial\nu_j$, where $f^{(j)}$ is the complex correspondent of the coefficient of δq_j in the virtual work done by the forces considered. The phenomenon of resonance under these forces is discussed, as is the case where p linear restrictions $\sum_k \alpha_j{}^k\nu_k=0$ are imposed on (S).

A system (S) with but one parameter q is called an impedence. The quadratic impedance Z of an impedence has the form $Z=z\nu^2$; the complex numbers z and $1/z$ are called the impedance and the admittance of the system, respectively. A set of n impedances (z_i) for which the velocities ν_i are all equal is said to be in series; n such in series form a new impedance (z) whose impedence z is the sum of the component impedences: $z=z_1+\cdots+z_n$. A somewhat more complicated way of combining n impedences in parallel is discussed, for which the admittances are additive: $1/z=1/z_1+\cdots+1/z_n$. It is pointed out how T, V, S may be interpreted to yield familiar theorems of electromagnetic theory. For the rest a discussion of applications to special mechanical spinning systems under viscous resistance is given. The paper contains no proofs. *A. L. Foster.*

Seth, B. R. On the gravest mode of some vibrating systems. Proc. Indian Acad. Sci., Sect. A. 13, 390–394 (1941). [MF 5092]

The author remarks that the gravest mode of any vibrating system should be devoid of internal nodal lines. In certain well-known cases only the symmetrical solutions have been obtained and hence the gravest modes remain undetermined. The determination of the assymetrical solutions, and hence the gravest modes, gives rise to an infinite determinant for the frequency. Rayleigh's method can be used to obtain an approximate value of the gravest mode. This is illustrated by discussing the transverse oscillations of water contained in a canal whose sides are inclined at 60° to the vertical. *A. E. Heins* (Lafayette, Ind.).

Roe, Glenn M. Frequency distribution of normal modes. J. Acoust. Soc. Amer. 13, 1–7 (1941). [MF 4860]

The main interest of the paper is the approximation for small $\bar\nu$ of $N(\bar\nu)$, the number of characteristic values not greater than $\bar\nu$ for $\nabla^2\psi+(2\pi\nu/c)^2\psi=0$, with $\psi=0$ on the boundary. For instance, for the rectangular enclosure of sides L_i the characteristic values are

$$(c/2)(\textstyle\sum_{i=1}^3 (n_i/L_i)^2)^{\frac12}-1, \qquad n_i=0,1,\cdots,$$

and $N(\bar\nu)$ is essentially

$$\textstyle\sum_{n_1=0}^{2\bar\nu L_1/c}\ (\sum_{n_2=0}^{n_2'}[1+(L_3/L_2)(n_2'^2-n_2^2)^{\frac12}])-1,$$

where n_2' is the nearest integer to $[(2\pi L_2/c)^2-n_1^2]^{\frac12}$. The sums are evaluated by expanding $(n_2'^2-n_2^2)^{\frac12}$ in a power series and rearranging terms. The known result is obtained, namely

$$N(\bar\nu)\sim 4\pi V\left(\frac{\nu}{c}\right)^3+\pi A\left(\frac{\nu}{2c}\right)^2+\frac{L\nu}{2c}, \tag{1}$$

where V is the volume, A the surface area and $L=\sum L_i$. The implied novelty in this paper is (a) the use of the Euler formula for $\sum_{n=1}^N n^k$ and (b) extension of this sort of computation to the cylinder and sphere and certain sections of these configurations. [However, apart from references to asymptotic results, the extensive mathematical literature on this Gitterpunkt problem is not mentioned.] The first two terms in (1) are the same for the cylinder and sphere also. The paper concludes with a derivation of sound decay in a cylinder following the calculation for rectangular enclosures given by P. M. Morse [Vibration and Sound, McGraw Hill, New York]. *D. G. Bourgin* (Urbana, Ill.).

Rainich, G. Y. Conditional invariants. Proc. Nat. Acad. Sci. U. S. A. 27, 352–355 (1941). [MF 4853]

Rainich, G. Y. The Dirac equations and conditional invariants. Proc. Nat. Acad. Sci. U. S. A. 27, 355–358 (1941). [MF 4854]

The notion of conditional invariants introduced in the first of these papers is used in the second to discuss the tensor form of the Dirac equations. The results have been previously obtained by Whittaker [Proc. Roy. Soc. London, Ser. A. 158, 38–46 (1937)]. The tensor form of the Dirac equations involving what the author calls "Darwin quantities" have been obtained in the general case by Ruse [Proc. Roy. Soc. Edinburgh 57, part 2, 97–127 (1936–37)]. *A. H. Taub* (Seattle, Wash.).

Chaundy, T. W. Systems of total differential equations. Quart. J. Math., Oxford Ser. 12, 61–64 (1941). [MF 4673]

Dès 1879 H. W. L. Tanner avait signalé que le premier membre de la condition d'intégrabilité de l'équation $Pdx+Qdy+Rdz=0$ peut se représenter symboliquement par le déterminant

$$\begin{vmatrix} P & \partial/\partial x & P \\ Q & \partial/\partial y & Q \\ R & \partial/\partial z & R \end{vmatrix}.$$

L'Auteur généralise cette expression pour exprimer la condition afin qu' existe un système de fonctions $u_1, u_2, \cdots, u_m$

tel que le système $du_r=0$ ($r=1, 2, \cdots, m$) soit équivalent au système $\sum_s P_{rs}dx_s=0$. Il faut et il suffit que soient nuls tous les déterminants d'ordre $m+2$ extraits des m matrices symboliques

$$\left\| P_{1s}P_{2s}\cdots P_{ms}\frac{\partial}{\partial x_s}P_{rs}\right\|.$$

L'Auteur expose la démonstration de la nécessité, suffisance et invariance de cette condition par rapport aux changements de variables sur le cas particulier de deux equations et quatre variables. *B. Levi* (Rosario).

Ostrowski, Alexandre. Sur une classe de transformations différentielles dans l'espace à trois dimensions. I. Comment. Math. Helv. 13, 156–194 (1941).

If $y_i(x)$, $i=1, \cdots, n$, are n indeterminate functions of a single variable x, and p_i are the derivatives of y_i, there exist transformations

$$(1) \qquad \xi=\xi(x, y_i, p_i), \quad \eta_j=\eta_j(x, y_i, p_i), \qquad j=1, \cdots, n,$$

such that

$$(2) \qquad x=x(\xi, \eta_i, \pi_i), \quad y_j=y_j(\xi, \eta_i, \pi_i),$$

where $\pi_i=d\eta_i/d\xi$. Such transformations are called by the author R transformations. The present paper is devoted to the case $n=2$. Some of his results are: Contact transformations in the sense of Sophus Lie can not exist for $n=2$ if any p_i or π_i are present in (1) or (2). Associated with each R transformation are two functions $r(x, y_i, p_i)$ and $\rho(\xi, \eta_i, \pi_i)$ such that the right sides of (1) can be expressed in terms of x, y_1, y_2, r, and the right sides of (2) in terms of $\xi, \eta_2, \eta_2, \rho$. The function r can be canonically defined by a Pfaffian equation $ds=0$, and ρ by a Pfaffian equation $d\sigma=0$. Then the R transformation can be thought of as a point transformation of the spaces x, y_1, y_2, r and $\xi, \eta_1, \eta_2, \rho$ satisfying the condition $d\sigma=\mu ds$. *F. G. Dressel* (Durham, N. C.).

Sintsov, D. M. Recherches sur les variétés Pfaffiennes. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 62–81 (1940). (Russian. French summary) [MF 4732]

A historico-critical account of the theory of Pfaffian varieties, satisfying $Pdx+Qdy+Rdz=0$. The paper is expository although some of the proofs are improvements on their originals. *M. S. Knebelman* (Pullman, Wash.).

van der Kulk, W. Eine Verallgemeinerung eines Theorems aus der Theorie der Pfaff'schen Gleichungen für den einfachsten Fall $m=2$. I. Nederl. Akad. Wetensch., Proc. 44, 452–463 (1941). [MF 5923]

The equations for the pseudobivector $v^{\kappa\lambda}$

$$(\mathrm{A}) \quad v^{[\kappa\lambda}v^{\mu\nu]}=0, \quad \overset{i}{F}(\xi^\kappa, v^{\kappa\lambda})=0, \qquad i=d+1, \cdots, 2(n-2); \quad \kappa, \lambda, \mu, \cdots=1, 2, \cdots, n,$$

in which the $\overset{i}{F}$ are analytic functions in the neighborhood of a zero point $\underset{0}{\xi^\kappa}, \underset{0}{v^{\kappa\lambda}}\neq 0$ of (A) and homogeneous in $v^{\kappa\lambda}$, determine a so-called $\mathfrak{S}_d{}^2$-field in the projective space of all pseudobivectors at ξ^κ. An integral X_2 is a manifold of 2 dimensions for which the tangent pseudobivector field satisfies (A). For such an X_2 the linear equations

$$(\mathrm{B}) \qquad v^{\mu\omega}\{\partial_\omega \overset{i}{F}+\overset{i}{F}_{\kappa\mu}v^{\lambda\mu}Z^\kappa_{\omega\lambda}\}=0,$$

in which $Z^\kappa_{\omega\lambda}=Z^\kappa_{\lambda\omega}$ are unknown and

$$\partial_\omega \overset{i}{F}=\partial \overset{i}{F}/\partial\xi^\omega, \quad \overset{i}{F}_{\mu\lambda}=\partial \overset{i}{F}/\partial v^{\mu\lambda},$$

have at least one solution. The $\mathfrak{S}_d{}^2$-field is called complete if (B), for every system ξ^κ, $v^{\kappa\lambda}\neq 0$ satisfying (9), have at least one solution.

To the $\mathfrak{S}_d{}^2$-field belongs an $\mathfrak{R}_t$-field formed by all directions v^κ contained in the 2-directions belonging to the points $v^{\kappa\lambda}$ of the local $\mathfrak{S}_d{}^2$ in ξ^κ. The local $\mathfrak{R}_t$ is a ruled surface, which is developable if its tangent space does not change along a generator. The main theorem announced in this paper is now as follows. A complete $\mathfrak{S}_d{}^2$-field, of which the $\mathfrak{R}_t$-field is developable, is completely integrable. This is a generalization of results by Cartan [Ann. École Norm. 18, 241–341 (1901)] and Kähler [Hamburg. Math. Einzelschr. 16 (1934)]. *D. J. Struik* (Cambridge, Mass.).

van der Kulk, W. Eine Verallgemeinerung eines Theorems aus der Theorie der Pfaff'schen Gleichungen für den einfachsten Fall $m=2$. II. Nederl. Akad. Wetensch., Proc. 44, 625–635 (1941). [MF 4982]

This second paper contains the proof of the theorem announced and prepared in the first paper. There are here and there remarks pointing to the case of integral x_m, $m>2$. *D. J. Struik* (Cambridge, Mass.).

Auerbach, H. Sur la parenthèse de Jacobi. Rec. Math. [Mat. Sbornik] N. S. 9(51), 731–734 (1941). (French. Russian summary) [MF 5508]

Proof of the theorem that a common integral f of the equations

$$Af=\sum_i a_i(x_1, \cdots, x_n)\partial f/\partial x_i=0,$$
$$Bf=\sum_i b_i(x_1, \cdots, x_n)\partial f/\partial x_i=0$$

is also a solution of the equation

$$ABf-BAf=\sum_i (Ab_i-Ba_i)\partial f/\partial x_i=0$$

for the case that the a_i, b_i and f have continuous derivatives of the first order. In this case the expression $AB-BA$ may no longer have a meaning. The existence of the theorem for this case has, as the author observes, already been established by E. Schmidt [Monatsh. Math. Phys. 48, 426–432 (1939); cf. these Rev. 1, 76] but his demonstration is different. *D. J. Struik* (Cambridge, Mass.).

Mendes, Marcel. Sur les systèmes d'équations du premier ordre en involution. C. R. Acad. Sci. Paris 211, 58–59 (1940). [MF 5334]

Let the functional determinant

$$\Delta=D(F_1, \cdots, F_n)/D(p_1, \cdots, p_n)$$

of the system of partial differential equations in involution

$$(*) \qquad F_i(x_k, p_j)=0, \qquad p_i=\partial z/\partial x_i, \quad i, j, k=1, \cdots, n,$$

vanish at the point $A(x_k{}^0, p_j{}^0)$ without being identically zero at that point. The author states results for two cases: one, if the F_i are linear in the p_i then (*) has a solution which is holomorphic at point A; and, two, if a minor of order $n-1$ of Δ does not vanish and the p_i from (*) take the form $p_i=P_i+Q_i\sqrt{R}$, then the solution of (*) can be put in the form $z=H+TR^{\frac{1}{2}}$. The functions P_i, Q_i, R, H, T are holo-

morphic, and R is assumed to vanish at point A and to contain a term of the first degree. *F. G. Dressel.*

Picone, Mauro. Problemi riducibili d'integrazione delle equazioni lineari a derivate parziali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 642–655 (1940). [MF 4991]

Making use of Green's functions, the author shows that certain rather general linear partial differential equations with preassigned boundary values can be reduced to the solution of a linear integral equation. *F. G. Dressel.*

Picone, Mauro. Nuovi metodi risolutivi per i problemi d'integrazione delle equazioni lineari a derivate parziali e nuova applicazione della trasformata multipla di Laplace nel caso delle equazioni a coefficienti costanti. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 413–426 (1940). [MF 4783]

Let

$$(1)\qquad E(u)=\sum_{k_1+\cdots+k_r\leq n} a_{k_1,\cdots,k_r}(X)\frac{\partial^{k_1+\cdots+k_r}u}{\partial x^{k_1}\cdots\partial x^{k_r}}=f,$$

where $X=(x_1,\cdots,x_r)$. Let $E^*(u)$ be the adjoint expression. If the domain D with boundary FD and the functions f and $a_{k_1,\ldots,k_r}$ are properly restricted, Green's formula applies in the form

$$\int_D uE^*(v)=\int_D f+\int_{FD}\sum U_i(u)E_i(v),$$

where $E_i(v)$ is linear in partial derivatives of order i. The writer observes that if U_i, $i=0,\cdots,n-1$, is given on FD then a solution of (1) is determined and may be obtained by selecting a sequence $\{v_s\}$ such that $\{\phi_s|\phi_s=E^*(v_s)\}$ is complete on D (for functions in C^n in the interior of D and in C^{n-1} on FD); namely

$$\int_D u\phi_s=\int fv_s+\int\sum U_iE_i(v_s).$$

If a sequence w_s of solutions of $E^*(v)=0$ exists, satisfying certain completeness properties, not all U_i are independent. If $\{E(w_{s,i})\}$, $i=1,\cdots,n$; $s=1,2,\cdots$, forms a complete system of "n vectors," then U_i need be given on part, only, of FD. If $a_{k_1,\ldots,k_r}$ is constant, take $v=\exp(-\sum\xi_i x_i)$. Suppose

$$u'=\int_0^\infty u\exp(-\sum\xi_i x_i)dX$$

and f' is similarly defined as the Laplace transform of f and let $P=\sum a_{k_1,\ldots,k_r}\xi^{k_1}\cdots\xi^{k_r}$. Then

$$(2)\qquad Pu'=f'+\int_{FD}\sum U_iE_i\exp(-\sum\xi_i x_i).$$

A necessary condition for the existence of a solution u of (1) is that the right side of (2) be divisible by P and, if P may be expressed as the product of linear factors, this divisibility property determines a solution (which the writer gets by using a suitable choice for v_s defined above). Remarks similar to those in the last three sentences for $r=1$ and $r=2$, as well as a generalization to the non-linear case, are contained in Bourgin-Duffin [Bull. Amer. Math. Soc. **45**, 859–869 (1939); cf. these Rev. **1**, 180].
D. G. Bourgin (Urbana, Ill.).

Michlin, S. Application de la transformation de Laplace aux problèmes limites pour l'équation des ondes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 305–307 (1941). [MF 4834]

The problem of solution of the wave equation in two dimensions for a cylinder above $t=0$ with directrix L and Cauchy data on the base and Dirichlet data on the wall has been previously shown by the author to be tantamount to the determination of $Q(M)$ from

$$Q(M_0)-\frac{1}{\pi}\iint_{\sigma_0}\frac{\partial Q(M)}{\partial t}\frac{(t_0-t)\cos(n,r)}{r[(t-t_0)^2-r^2]^{\frac12}}dS=A(M_0)$$

[the same C. R. **29**, 281–285 (1940); cf. these Rev. **2**, 291]. Here σ_0 is the region cut out on the cylinder wall by the characteristic cone with vertex on the cylinder and $A(M_0)$ is a known function. The conventional device is followed of taking the Laplace transform of both equations, solving the transformed equation for the transform of Q and then applying the Mellin transform to obtain Q. This requires, of course, existence of a solution of the transformed integral equation which is, for instance, analytic in a half plane, etc., and a partial justification is sketched by the author. Essentially similar methods may be applied to the wave equation in three dimensions. *D. G. Bourgin* (Urbana, Ill.).

Aronszajn, Nathan and Weinstein, Alexander. Existence, convergence and equivalence in the unified theory of eigenvalues of plates and membranes. Proc. Nat. Acad. Sci. U. S. A. **27**, 188–191 (1941). [MF 3903]

Weinstein has made a significant contribution to the theory of vibration of clamped plates by recognizing that this problem can be interpreted as the limiting case of a sequence of problems of second order referring to the differential equation of the membrane and formulated by variational problems with an increasing number of subsidiary conditions. The present note is a brief report on the more subtle reasonings necessary for the existence proofs and more so for the proof that the solutions of the approximating problems actually converge to the solution of the plate problem. *R. Courant* (New York, N. Y.).

***Siddiqi, M. R. Boundary Problems in Non-linear Partial Differential Equations.** Lucknow University Studies, no. 11. Allahabad Law Journal Press, Allahabad, India, 1939. xiv+136 pp.

In a sequence of papers the author has developed, since 1932, a theory of certain types of non-linear parabolic and hyperbolic differential equations with two independent variables. The essential procedure is an expansion of the solution in a series by eigenfunctions of a naturally associated linear problem; the coefficients of the expansion are then determined by an infinite system of non-linear integral equations, which in turn can be attacked by an iteration method. Lichtenstein had first suggested the pattern of such a treatment in his work on the differential equation $u_{xx}-u_{tt}=u^3$, where an expansion $u(x,t)=\sum v_n(t)\sin nx$ is discussed. Siddiqi's papers have extended Lichtenstein's theory to a much wider class of problems taking into account also the work by Picard, Holmgren, E. E. Levi and Gevrey which is based on the use of Green's function or similar devices. The present little book gives a simplified, enlarged and improved presentation of the author's contributions including an analysis of the integro-differential

equations involved. The emphasis is on existence and uniqueness proofs for rectangular and more general domains.
R. Courant (New York, N. Y.).

Thiruvenkatachar, V. R. Solution of boundary value problems by double Laplace transformations. J. Mysore Univ. Sect. B. 1, 115–121 (1941). [MF 5140]

The procedure of solving boundary value problems by using iterated Laplace transforms with respect to two independent variables is illustrated further by solving four temperature problems in the one-dimensional flow of heat in a semi-infinite medium. *R. V. Churchill.*

Malkin, I. On the problem of temperature distribution in plane plates. J. Franklin Inst. **232**, 129–150 (1941). [MF 5006]

Let coordinates be chosen so that the faces of a thin plate lie in the planes $z=\pm 1$. Simple closed forms are sought which approximate the temperature function for the plate when the boundary values are continuous. In the steady state case the following form of the temperature function is used:

$$T(x,y,z)=\sum_{i=0,1,2,\cdots}[F_i(z)D^iT^0(x,y)+G_i(z)D^iP^0(x,y)],$$

where $D=\partial^2/\partial x^2+\partial^2/\partial y^2$ and T^0 and P^0 are polynomials in x and y. Thus $\partial^2T/\partial z^2+DT=0$ provided that $F_0''(z)=0$, $F_{i+1}''(z)+F_i(z)=0$, $G_0''(z)=0$, and that $G_{i+1}''(z)+G_i(z)=0$. Polynomial expressions for the functions F_i, G_i, T^0 and P^0 are easily found such that $T(x,y,\pm 1)$ are any prescribed polynomials in x and y. The temperature distribution over the edge of the plate is fixed by the solution found; but it is shown that these edge temperatures may be adjusted in some cases by the superposition of another simple temperature function. The procedure is essentially the same when cylindrical coordinates are used, and when radiation takes place at the faces. Also, the method is extended so as to apply to temperatures that vary with the time. Numerical examples are included in which the variable and steady state temperatures are found for circular plates.
R. V. Churchill (Ann Arbor, Mich.).

Lidjaev, S. Über die Darstellbarkeit der Lösung der Wärmeleitungsgleichung durch das Poissonsche Integral. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 263–268 (1941). (Russian. German summary) [MF 5177]

The author shows by straightforward reasoning that the Poisson integral (both limits infinite) corresponding to the heat equation $u_{xx}=u_t$, which converges for one particular (x_0, t_0), has, with respect to convergence to the boundary values, the same behavior as the Poisson integral for the Laplace equation. He also gives necessary and sufficient conditions for a solution of the heat equation to be representable by a Poisson or by a Poisson-Stieltjes integral. These conditions are of the type: $u(x,t)>0$, $|u(x,t)|e^{-cx^2}<K/\sqrt{t}$, for $t<t_0$, and convergence of $\int_{-\infty}^{\infty}e^{-cx^2}u(x,t)dx$ uniformly for $t<t_0$. *F. Wolf* (St. Paul, Minn.).

Sakadi, Zyuro. On the disturbance of flow of air by heat conduction from the ground. Proc. Phys.-Math. Soc. Japan (3) **23**, 214–226 (1941). [MF 4524]

The problem is that of the solution of the kinetic theory transport equations for first order increments in the velocities when there is a small constant mass velocity U in the x_1 direction and gravity is neglected. If x_1 and x_3 are coordinates in the plane earth surface and x_2 is the vertical coordinate, the mathematical problem reduces to satisfying (1) $\Delta\chi=\sigma^2\chi$ and (2) $\Delta\psi=-(U/2\sigma)\Delta(e^{\sigma x_1}\chi)$ subject to (a) $\partial\psi/\partial x_2|_{x_2=0}=0$ and (b) $e^{\sigma x_1}\chi|_{x_2=0}=\phi(x_1,x_3)$, where ψ is the velocity potential sought for, $e^{\sigma x_1}\chi T_0$ is the increment in temperature $T-T_0$ and σ is a certain constant. [Condition (b) involves somewhat dubious approximations for the actual physical situation under consideration.] The author eventually takes ϕ as the Dirac function and then (1) and (b) yield $\chi=c_1\partial(e^{-\sigma R}/R)/\partial x_2$. Then the potential function $(\psi+(U/2\sigma)e^{\sigma x_1}\chi)$, vanishing for $x_2\to-\infty$, is determined by (a). Since this is a standard type of Neumann problem the rigorous solution can be written down at once but the author makes no reference to known results and a large part of the paper is devoted to rigorously establishing the validity of the limiting processes involved in obtaining this potential function. For sufficiently large σ, R and x_2 the representation of ψ may be adequately approximated by a simple expression. *D. G. Bourgin* (Urbana, Ill.).

Vecoua, N. Über den Grenzübergang von den dynamischen Prozessen zu den stationären in Randproblemen der Wärmeleitung. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 651–657 (1940). (Russian. German summary) [MF 5286]

Let C be a closed curve in the xy plane, $r_{\sigma s}$ the distance between points s and σ on C, and n_σ the normal to C at point σ. Also let $\varphi^*(s,t)$ be the solution of the integral equation

$$(1)\quad \int_C r_{\sigma s}\cos(r_{\sigma s},n_\sigma)d\sigma\int_0^t\frac{\varphi(\sigma,v)}{4k(t-v)^2}\exp\left(-\frac{r_{\sigma s}^2}{4k(t-v)}\right)dv=f(s,t)-\pi\varphi(s,t),$$

where the given continuous function $f(s,t)$ has a continuous limit function $f(s)$ as t becomes infinite. The object of the paper is to show that as $t\to\infty$ the limit of $\varphi^*(s,t)$ exists and is the solution of the integral equation

$$(2)\qquad \pi\varphi(s)+\int_C\varphi(\sigma)\cos(r_{\sigma s},n_\sigma)d\sigma=f(s).$$

Equation (1) arises in solving an interior Dirichlet problem for the equation $k(u_{xx}+u_{yy})=u_t$ [C. Müntz, Math. Z. **38**, 323–337 (1933–34)], while equation (2) arises in solving the interior Dirichlet problem for the equation $u_{xx}+u_{yy}=0$.
F. G. Dressel (Durham, N. C.).

Vecoua, Elias. Über harmonische und metaharmonische Funktionen im Raum. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 29–34 (1941). (German. Georgian summary) [MF 5295]

The author proves the following theorem: let $\omega(r,\varphi,\theta)$ be a harmonic function in a region T, star-shaped with respect to the origin, with possibly a singularity of type $1/r$ at the origin. Define

$$(1)\quad u(r,\phi,\theta)=\omega(r,\phi,\theta)-\int_{\rho=0}^{r}(\lambda/2)\rho^{\frac12}(r-\rho)^{-\frac12}J_1(\lambda(r(r-\rho))^{\frac12})\omega(\rho,\phi,\theta)d\rho,$$

where J_1 is the first Bessel function. Then u is metaharmonic, that is, satisfies $\Delta u+\lambda^2u=0$, with a possible singularity of type $1/r$ at the origin. Conversely, if u is given as such a meta-harmonic function, it can be expressed in exactly one way in the form (1). The proof of the first part

of the theorem is made by formal means. The converse is obtained by considering (1) as an integral equation of Volterra type for ω. *J. W. Green* (Rochester, N. Y.).

Grünberg, G. Über die Kurzschlusserwärmung von Hochspannungskabeln. Acad. Sci. USSR. J. Phys. **4**, 463–472 (1941). [MF 4856]

A solution to the problem of heating a high tension oil cable by short circuit is given under the assumption that one can neglect the displacement of the oil throughout the process. Mathematically the problem may be stated as follows. Given three concentric cylinders with different materials between each set of boundaries. In the inner cylinder and the outer ring the usual equation of heat conduction in polar coordinates is satisfied, while in the center ring a modified form of the equation is satisfied. The problem is to determine the temperature distribution in the middle ring, having specified initial and boundary conditions for each medium. This is accomplished by transforming over the time variable with the Laplace transform and by introducing certain reasonable simplifications into the differential equations. *A. E. Heins.*

Weinberg, Alvin M. Weber's theory of the Kernleiter. Bull. Math. Biophys. **3**, 39–55 (1941). [MF 4485]

The author studies the radial distribution of currents in a semi-infinite conductor of radius b, covered with a membrane and surrounded by a cylinder of radius $a>b$, the intervening medium being an electrolyte. Assuming radial symmetry, the problem reduces to that of finding a function $u(r, z)$ which is harmonic in $r>0$, $z>0$, has given initial values for $z=0$, vanishes as $z\to\infty$ and which for $r=a$ and $r=b$ satisfies the conditions $\partial u/\partial r=0$ and $h\partial u/\partial r=u$, respectively. The author studies in detail H. Weber's classical solution of this problem and some of its experimental implications. The solution is represented by an infinite series of Bessel functions the arguments of which depend on the solutions of a characteristic equation; the author indicates how its roots may be approximated. *W. Feller.*

Vestine, E. H. On the analysis of surface magnetic fields by integrals. I. Terr. Magnetism **46**, 27–41 (1941). [MF 4644]

This paper forms Part 1 of a series presenting a general method of analyzing surface geomagnetic fields by means of surface integrals. In geophysical applications the method is free from some of the limitations usual to spherical harmonic analysis. The main problem discussed in the paper may be posed as follows: Consider a region v enclosed by a surface S in which there are bodies of magnetic matter M_i. Suppose, also, that there are bodies of magnetic matter M_e in the region outside S. Let V_i and V_e be the magnetic potentials due to M_i and M_e, respectively, their combined potentials being equal to V on S. It is required to determine the values of V_i and V_e separately from a knowledge of V and $\delta V/\delta n$ on S. The solution of this problem is based on Green's theorem and subsidiary results. Relations between the potential and its space derivatives in the immediate neighborhood of a closed surface are developed, especially for spherical and plane surfaces. *M. A. Basoco.*

Zeragin, P. K. On the integration of polyharmonic equations. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **8**, 135–163 (1940). (Georgian. Russian summary) [MF 5313]

Reade, Maxwell and Beckenbach, E. F. An integral analogue of Laplace's equation. Bull. Amer. Math. Soc. **47**, 633–640 (1941). [MF 5059]

The authors obtain an integral analogue of Laplace's equation, related to Laplace's equation as Morera's theorem is related to the Cauchy-Riemann differential equations. Corresponding to the fact that Laplace's equation is of the second order, their integral analogue involves a double integral. Their first theorem is: If the real function $x(u, v)$ is continuous in a finite simply connected domain D, then a necessary and sufficient condition that $x(u, v)$ be harmonic in D is that

$$\int_{C^*}\left[\int_{C(s,t)} x(s+\xi, t+\eta)(d\xi-id\eta)\right](ds+idt)=0$$

hold for all curve pairs (C, C^*) for which the curves $C(s, t)$ all lie in D. Here C, C^* are closed rectifiable Jordan curves C: $u=s(\tau)$, $v=t(\tau)$, $0\leqq\tau\leqq1$; C^*: $u=\xi(\tau)$, $v=\eta(\tau)$; and $C(s, t)$ for fixed s, t is the curve $u=s+\xi(\tau)$, $v=t+\eta(\tau)$, $0\leqq\tau\leqq1$. The necessity proof uses Green's lemma and Cauchy's theorem, the sufficiency proof a simple averaging process and Morera's theorem. In both parts Laplace's equation is considered as an iterated equation $(\partial/\partial u+i\partial/\partial v)\times(\partial/\partial u-i\partial/\partial v)x=0$. In the next two theorems the sufficiency conditions are lessened somewhat.

W. T. Martin (Cambridge, Mass.).

Rosenblatt, Alfred. On Green's function of bounded domains in Euclidean space of three dimensions. Actas Acad. Ci. Lima **4**, 42–52 (1941). (Spanish) [MF 5227]

Let D be a finite domain in 3-dimensional Euclidean space, bounded by a closed surface S, with Green's function $G(P, P')$. The author [C. R. Acad. Sci. Paris **201**, 22–24 (1935)] previously has obtained an inequality for $G(P, P')$ on the assumption that S has a continuous curvature, namely $G(P, P')<K\delta\delta'/r^3$, where K is a positive constant depending on S, δ and δ' are the respective distances of P and P' from S and r is the distance PP'. Keldych and Lavrentieff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 102–103 (1939); these Rev. **2**, 57] subsequently used properties of Legendre polynomials to establish the same result for an arbitrary surface of Liapounoff. In the present paper the method of Keldych and Lavrentieff is applied to further improve the above result in two directions: the inequality is somewhat sharpened and the improved inequality shown to hold for the much broader class of surfaces S satisfying the condition that at each point M of S there is a sheet of a right circular cone of uniform vertical angle $\alpha>0$ and uniform altitude $h>0$ and having no point other than M in common with S. *E. F. Beckenbach.*

Gevrey, Maurice. Sur une généralisation du principe des singularités positives de M. Picard. C. R. Acad. Sci. Paris **211**, 581–584 (1940). [MF 5364]

Let $u(P)$ be a solution of Laplace's equation $u_{x_1x_1}+\cdots+u_{x_nx_n}=0$ for all points $P(x_1, \cdots, x_n)$ in the neighborhood of the origin O such that $u(P)\to+\infty$ as $P\to O$. Then the principle of positive singularities of Picard states that, for $n>2$, $u(P)=w\cdot r^{2-n}$. The function $w(P)$ is continuous and positive at the origin, and r is the distance OP. The author [C. R. Acad. Sci. Paris **183**, 544 (1926)] has previously stated that a similar result holds for solutions of the elliptic equation $\sum a_{ij}u_{x_ix_j}+\sum b_iu_{x_i}+cu=f$. The main object of the present paper is to show that the principle of positive

singularities also holds for solutions of the equation $Fu=f$, where F is the author's generalized elliptic operator [cf. Ann. École Norm. (3) **52**, 39–108 (1935), in particular, p. 57]. Besides Picard's condition on the behavior of $u(P)$ at the origin the author needs a condition of the type $u(P)=O(r^{2-n})$.

F. G. Dressel (Durham, N. C.).

Monna, A. F. Quelques applications de la théorie moderne du potentiel aux fonctions holomorphes. Nederl. Akad. Wetensch., Proc. **44**, 718–726 (1941). [MF 5118]

The portion of the modern potential theory which interests the author is the representation of harmonic and subharmonic functions by means of Stieltjes integrals, and he translates a number of these results into the corresponding representation theorems for analytic functions. If Ω is a domain with boundary Σ, $\mu(P, e)$ the harmonic measure with respect to Ω and f a bounded analytic function in Ω with zeros at a_i of order n_i, $\log|f|$ is subharmonic and admits the representation

$$\log|f| = -\sum_i n_i G(z, a_i) + \int_\Sigma \lambda(Q) d\mu(P, e_Q),$$

where G is the Green's function for Ω, $\lambda(Q)=\lim_{P\to Q} f(P)$ and f is required to be continuous on Σ p.p. with respect to $\mu(P, e)$. This gives a corresponding Stieltjes representation of f:

$$f(z)=\prod_i \exp\{-n_i[G(z, a_i)+i\bar{G}(z, a_i)]\}$$
$$\cdot\exp\int_\Sigma \log\lambda(Q)d[\mu(z, Q)+i\bar{\mu}(z, Q)],$$

where the bar denotes conjugate function. Also if $u(z)$ is a positive harmonic function in Ω, it admits, under certain conditions on Ω at least, a representation of the form $u(z)=\int_\Sigma \gamma(z, Q)d\theta(e_Q)$, where γ depends only on Ω and θ is a positive mass distribution depending on u. If this is applied to $-\log|f|$, a representation of the bounded analytic function f is obtained, which is well known in case Ω is the circle. The author does not impose any condition on Ω, which would seem to be necessary if the integral representation of $u(P)$ is always to be possible.

J. W. Green (Rochester, N. Y.).

Brelot, M. Points irréguliers et transformations continues en théorie du potentiel. J. Math. Pures Appl. (9) **19**, 319–337 (1940). [MF 4627]

A set E is "attenuated," "thin" [effilé] at a point O, belonging or not to E, if O is isolated from E or if there exists a distribution of positive mass in the neighborhood of O whose potential is less at O than its inferior limit for approach to O from E. If E is closed and O a frontier point of E, and E is attenuated at O, then O is an irregular point of E; if CE is attenuated at O, then O is a stable point of E [Brelot, Bull. Sci. Math. (2) **63**, 79–96, 115–128 (1939); cf. "exceptional point," Evans, Trans. Amer. Math. Soc. **37**, 226–253 (1935), especially p. 243]. The author gives detailed proofs of theorems stated in his previous note [C. R. Acad. Sci. Paris **209**, 828–830 (1939); these Rev. **1**, 121], and applies transformations which diminish homologous distances to the analysis of sets which are attenuated at a given point.

G. C. Evans (Berkeley, Calif.).

Integral Equations

Sato, Tunezo. On eigenvalues of iterated kernels. Proc. Phys.-Math. Soc. Japan (3) **23**, 4–7 (1941). [MF 3907]

In generalizing a well-known theorem the author proves that to each characteristic value λ of the polynomial kernel $f(K)=a_1K+a_2K^2+\cdots+a_nK^n$ (K^r is the rth iterated kernel) there exists at least one characteristic value α of K which is a root of the algebraic equation $f(\alpha^{-1})=\lambda^{-1}$. The proof is formally analogous to that of the particular case $f(K)=K^n$; to show that a non-vanishing characteristic function of K can be found, a somewhat different recurrent process is applied.

E. Hellinger (Evanston, Ill.).

Giraud, Georges. Équations de Fredholm dont le noyau est fonction holomorphe d'un paramètre. Bull. Sci. Math. (2) **64**, 225–244 (1940). [MF 5236]

Consider the homogeneous integral equation

$$u(X)=\int G(X, A; \lambda)u(A)dV_A, \tag{1}$$

where the integration is over an m dimensional region V and the kernel G is a holomorphic function of λ in a region D of the complex plane. If G has a resolvent, denote it by $N(X, A; \lambda)$; otherwise let N denote one of the pseudoresolvents to G. If $K\geqq 0$ is the minimum number of linearly independent solutions of (1) as λ varies in D, the author proves that at a pole of N in D the equation (1) has more than K linearly independent solutions. This is a direct generalization of the known case in which $K=0$ and N is the resolvent to G.

F. G. Dressel (Durham, N. C.).

Giraud, Georges. Équations de Fredholm dont le noyau est fonction holomorphe d'un paramètre; équations analogues où figurent des intégrales principales. C. R. Acad. Sci. Paris **211**, 47–49 (1940). [MF 5332]

This note states the results of the author's paper reviewed above. The following additional results are stated without proof. Let $g(X, \lambda)\neq 0$ and $G(X, Y; \lambda)$ be holomorphic functions of λ in the region D for points X and Y in the m dimensional region V. Let $K\geqq 0$ and $H\geqq 0$ be, respectively, the minimum number of linearly independent solutions of the integral equations (*) and (**) as λ varies in D.

$$g(X, \lambda)U(X)=\int_V G(X, Y; \lambda)U(Y)dV_Y; \tag{*}$$

$$g(X, \lambda)V(X)=\int_V G(Y, X; \lambda)V(Y)dV_Y. \tag{**}$$

Then a resolvent kernel $N(X, Y; \lambda)$ to these equations can be defined, such that in points λ where N is holomorphic equations (*) and (**) have, respectively, K and H linearly distinct solutions, and at a pole of N they have $K+T$ and $H+T$ ($T\geqq 1$) such solutions.

F. G. Dressel.

Smithies, F. The Fredholm theory of integral equations. Duke Math. J. **8**, 107–130 (1941).

Hilbert has shown in his original paper [1904] that Fredholm's formulas converge for some discontinuous kernels if the diagonal terms in the determinants are replaced by zeros. Carleman [Math. Z. **9**, 196 (1921)] extended this result to all kernels $k(s, t)$ for which $|k(s, t)|^2$ has a finite Lebesgue double integral. This result is also contained in Hille-Tamarkin's comprehensive theory of such integral equations, which is based on a reduction to equations

in infinitely many unknowns [Acta Math. 57, 1 (1931)]. Smithies approaches the problem in a new and slightly more general way which, however, is somewhat related to the last mentioned theory; he uses the language and the notions of the bounded linear transformations in a Hilbert space $\mathfrak{H}$. He investigates linear transformations "of finite norm" Kx in $\mathfrak{H}$, that is, such ones for which $\sum_{\alpha,\beta}|(Kx_\beta, x_\alpha)|^2$ converges, where $(x_1, x_2, \cdots)$ is an orthonormal complete set in $\mathfrak{H}$. First, he solves the equation $x-\lambda Kx=y$ in a finite-dimensional unitary space by expressions analogous to Fredholm's formulas, written in terms of the iterated transformations K^n and their traces $\tau(K^n)=\sum_\alpha(Kx_\alpha, x_\alpha)$. He modifies these formulas, as Poincaré has done, so that the first trace $\tau(K)$ drops out. Then he applies this to the projections of the general transformation (1) $x-\lambda Kx=y$ of $\mathfrak{H}$ into the p-dimensional spaces determined by $(x_1, \cdots, x_p)$. Since, under his suppositions, the traces $\tau(K^n)$ exist for $n\geqq 2$, he is able to prove that the inverse transformations of those projections converge to the inverse of (1); simultaneously he gets an analogous simple formula for the solution of (1). Eventually he can apply this result to Carleman's integral equation. *E. D. Hellinger.*

Vecoua, N. P. Integral equations of Volterra type with integrals in the sense of Hadamard. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 501–508 (1940). (Georgian. Russian summary) [MF 5282]

Kupradze, V. Zur Theorie der Integralgleichungen mit dem Integral im Sinne des Cauchyschen Hauptwertes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 255–262 (1941). (Russian. German summary) [MF 5176]

The author considers equations

$$(1)\quad K(\varphi(s))=a(s)\varphi(s)-\lambda b(s)\int_\gamma K(s,t)(t-s)^{-1}\varphi(t)dt=f(s),$$

where s and t are complex variables denoting points on the simple closed curve γ (with finite curvature); $a(s)$, $b(s)$, $K(s,t)$, $f(s)$ satisfy Hölder's condition; k and $\bar{k}$ denote the numbers φ_j, $\bar{\varphi}_j$ of distinct solutions of the two associated homogeneous equations $K(\varphi(s))=0$, $\bar{K}(\bar{\varphi}(s))=0$, respectively; $2\pi i n=\int_\gamma d\log(\alpha(s)-\beta(s))$, where $\alpha(s)=a(s)+\lambda\pi i A(s)$, $\beta(s)=a(s)-\lambda\pi i A(s)$. It is proved anew that (1) is soluble if and only if $\int_\gamma f(s)\bar{\varphi}_i(s)ds=0$ $(i=1,\cdots,\bar{k})$ and that $k-\bar{k}=2n$. The proof of the latter result involves methods of the type used by T. Carleman in his theory of singular integral equations. According to Mikhlin, (1) is reducible to an equivalent Fredholm equation if and only if $n\geqq 0$.

W. J. Trjitzinsky (Urbana, Ill.).

Miranda, Carlo. Nuovi contributi alla teoria delle equazioni integrali lineari con nucleo dipendente dal parametro. Mem. Accad. Sci. Torino (2) 70, 23–51 (1940). [MF 4995]

This paper deals with the development of the Hilbert-Schmidt theory for a linear integral equation of the form

$$\phi(x)=f(x)+\lambda\int_T G(x,y,\lambda)\phi(y)dy,$$

where the symmetric kernel $G(x,y,\lambda)$ is of the form

$$K(x,y)-\sum_{i=1}^{\infty}\frac{\lambda}{\lambda-a_i}H_i(x,y).$$

Various general properties of such an equation are established under the following hypotheses: (i) the constants a_i are real and distinct, and such that the series $\sum_i 1/|a_i|$ is convergent; (ii) the kernels $H_i(x,y)$, $K(x,y)$ are symmetric, $H_i(x,y)$ $(i=1,2,\cdots)$ are continuous and $K(x,y)$ is of integrable square on $T^{(2)}$; moreover, there exists a function $M(x,y)$ which is of integrable square on $T^{(2)}$, has $M(x,x)$ integrable on T, and such that $|H_i(x,y)|\leqq M(x,y)$ $(i=1,2,\cdots)$; (iii) the kernels $a_iH_i(x,y)$ $(i=1,2,\cdots)$ are positive semi-definite. Under the additional hypothesis that each $H_i(x,y)$ is of finite order the author establishes certain extremizing properties of the characteristic solutions and various expansion theorems in terms of the characteristic solutions. The results of this paper extend the previous work of the author [Rend. Circ. Mat. Palermo 60, 286–304 (1936)], Iglisch [Math. Ann. 117, 129–139 (1939); cf. these Rev. 1, 238] and Manià [Ann. Scuola Norm. Super. Pisa 8, 89–104 (1939)]. *W. T. Reid* (Chicago, Ill.).

Miranda, Carlo. Sulle equazioni integrali il cui nucleo è funzione lineare del parametro. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 7 pp. (1940). [MF 4989]

By methods analogous to those employed in the paper reviewed above, the author here establishes results on the existence and properties of characteristic values and functions, and associated expansion theorems, for the linear integral equation

$$\phi(x)=\lambda\int_T[H(x,y)+\lambda H_1(x,y)]\phi(y)dy$$

under the following hypotheses: on $T^{(2)}$ the kernels $H(x,y)$ and $H_1(x,y)$ are real and symmetric, $H(x,y)$ is of summable square, while $H_1(x,y)$ is a continuous and positive semi-definite kernel. *W. T. Reid* (Chicago, Ill.).

Caligo, Domenico. Un criterio sufficiente di stabilità per le soluzioni dei sistemi di equazioni integrali lineari e sue applicazioni. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 1, 497–506 (1940). [MF 4990]

The author establishes a sufficient condition for the stability of the solutions of a system of linear integral equations of Volterra type, and with the aid of this criterion deduces various sufficient conditions for the stability of the solutions of linear differential equations. *W. T. Reid.*

Poncin, Henri. Étude d'une équation intégrale de l'hydrodynamique du fluide visqueux (écoulement laminaire en régime variable). J. Math. Pures Appl. (9) 19, 163–195 (1940). [MF 4619]

The problem of free flow in a viscosimeter is found to depend on the solution of the integral equation

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{\partial w}{\partial t}=C+\int\int w(r,t)rdrdt.$$

The problem is made more precise by the consideration of an operator

$$A_{\alpha,\lambda}[f(n,t)]=\frac{\partial}{\partial n}\left(n\frac{\partial f}{\partial n}\right)-\alpha^2\frac{\partial f}{\partial t}-\lambda\int_0^1\int_0^t f(n,t)dndt$$

and the integro-differential equation $A_{\alpha,\lambda}[f(n,t)]+\beta=0$. In §1 the operator is applied to a function of type $f(n,t)=e^{-n^2t}f_i(n)$, and it is found that a useful set of functions $f_i(n)$ is obtained by writing

$$x_i^4J_0(x_i)f_i(n)=\mu[J_0(x_i\sqrt{n})-J_0(x_i)],$$

where $\mu=16a^2\lambda$ and x_i is one of the quantities $x(\mu)=2am$ defined by the transcendental equation

$$2\mu J_1(x)-x(x^4+\mu)J_0(x)=0.$$

A study of the function $x(\mu)$ is made in §2 and a 4-place table is given of μ as a function of $x_1(\mu)$ for $x_1(\mu)=\cdot 2(\cdot 1)1$ and for the values $2\cdot 10^{-2}$, $4\cdot 10^{-2}$, 10^{-1}. In the subsequent work use is made of asymptotic expressions and infinite series. *H. Bateman* (Pasadena, Calif.).

Lichnerowicz, André et Marrot, Raymond. Sur l'équation intégrodifférentielle de Boltzmann. C. R. Acad. Sci. Paris **211**, 531–533 (1940). [MF 5361]

This note supplements Carleman's memoir [Acta Math. 60, 91–146 (1933)] by replacing Carleman's assumption

$$0\leqq f_0(r)<a/(1+r)^\chi, \qquad \chi>6,$$

by the assumption of the convergence of the integral

$$\int_0^\infty |F|r^\nu dr, \qquad \nu>5.$$

B. O. Koopman (New York, N. Y.).

Functional Analysis, Ergodic Theory

Smiley, M. F. Measurability and distributivity in the theory of lattices. Bull. Amer. Math. Soc. **47**, 604–611 (1941). [MF 5053]

The author defines "strongly measurable" (s.m.) elements of a modular lattice with functional $\mu(x)$ as elements a which satisfy, for all b, c, the identity

$$2[\mu(a\cup b\cup c)-\mu(a\cap b\cap c)]=\mu(a\cup b)-\mu(a\cap b)+\cdots+\mu(b\cup c)-\mu(b\cap c).$$

He shows that, if μ is a positive functional, then the set of s.m. elements is a sublattice, which is topologically closed if the functional is continuous. If μ is strictly positive, then a complement of any s.m. element is measurable (in the sense of Carathéodory) if and only if it is a unique complement, and then it is s.m. Examples show the hypotheses are not redundant. Finally, earlier results of V. Glivenko [Amer. J. Math. 59, 941–956 (1937)] are given fresh interpretations. *G. Birkhoff* (Cambridge, Mass.).

Grunblum, M. M. Certains théorèmes sur la base dans un espace du type (*B*). C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 428–432 (1941). [MF 4838]

A Banach theorem states that a biorthogonal series is a basis if and only if the partial sums are uniformly bounded. This note gives a geometrical equivalent of this statement and also one involving linear functionals. *F. J. Murray.*

Krein, M., Milman, D. and Rutman, M. A note on basis in Banach space. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 106–110 (1940). (Russian. English summary) [MF 4736]

The following theorems are proved: (1) If $\{x_i\}$ is a basis for a Banach space E, there exists a sequence of $\delta_i>0$ such that $\{y_i\}$ is also a basis if $\|x_i-y_i\|<\delta_i$. (2) Let E be a space having a basis. Then, for any fundamental set $\{h_n\}\subset E$, there exists a basis $\{y_n\}$ having the form $y_n=\sum_{i=1}^{p_n}c_{ni}h_i$. (3) If $\{x_i\}\subset E$, $\{f_i\}\subset \bar{E}$ is a biorthogonal set, there exist $\delta_i>0$ such that $\|x_i-y_i\|<\delta_i$ implies the existence of $\{\varphi_i\}\subset\bar{E}$ such that $\{y_i\}$, $\{\varphi_i\}$ are biorthogonal. If $\{x_i\}$ is a fundamental set, the $\{\delta_i\}$ may be so chosen that $\{y_i\}$ is also a fundamental set. Theorem 2 applied to the space C with $h_n=t^n$ gives the following: There exists a sequence of polynomials $\{p_i(t)\}$ such that every continuous $x(t)$, $0\leqq t\leqq 1$, can be expressed in a unique manner by a uniformly convergent series $x(t)=\sum_{i=1}^\infty c_ip_i(t)$. The English summary is substantially a translation.

J. V. Wehausen (Columbia, Mo.).

Bohnenblust, F. Subspaces of $l_{p,n}$ spaces. Amer. J. Math. 63, 64–72 (1941). [MF 3630]

The purpose of this paper is to contribute towards the solution of the problem of whether every separable Banach space B admits a base. It is known that this question is closely connected with the properties of the sequence of numbers $a_m=\inf\|P\|$ ($m=1, 2, \cdots$), where $\|P\|$ is the norm of any linear projection of B on any subspace of given dimension m. The author finds that, even in classical spaces $L_{p,n}$ of elements $(x_1, \cdots, x_n)$ with the norm $\|x\|=[|x_1|^p+\cdots+|x_n|^p]^{1/p}$, sometimes l-dimensional subspaces exist with $a_m>1$ for $1<m<l$. The final theorem is that in a l-dimensional subspace S_l of $L_{p,n}$, where p is not an integer, only the identity and the projections on one-dimensional subspaces can have the norm one, provided S_l is "in general position" and $n>2(2l-3)$. The investigation is based on the introduction of Plücker-Grassmann coordinates which allow a thorough characterization and classification of all linear subspaces of $L_{p,n}$. *E. D. Hellinger.*

Pospíšil, Bedřich. Eine Bemerkung über vollständige Räume. Časopis Pěst. Mat. Fys. **70**, 38–41 (1940). (German. Czech summary) [MF 5427]

Let R denote a topologically complete space. The author seeks to determine the conditions under which this is true: (Z) If S is a countable subset of R such that every bounded continuous function defined on S can be extended to R, then S is closed in R. He finds that (Z) is true if R is completely normal, or if the cardinal number of R is less than the number of all sets of real numbers. *H. M. Gehman.*

Levine, B. and Milman, D. On linear sets in space C consisting of functions of bounded variation. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 102–105 (1940). (Russian. English summary) [MF 4735]

Let E be a linear closed subspace of C, the space of continuous functions on a bounded interval with

$$\|x\|=\max|x(t)|.$$

If E consists only of functions of bounded variation, it has finite dimension. The proof goes by way of the following lemma: If a sequence of bounded functions in E converges to zero at every point of the interval, it also converges uniformly. *J. V. Wehausen* (Columbia, Mo.).

Fortet, Robert. Les systèmes d'équations linéaires dans les espaces uniformément convexes. C. R. Acad. Sci. Paris **211**, 422–423 (1940). [MF 5357]

Let B be a uniformly convex vector space, $\{x_n\}$ a sequence in B with $\|x_n\|=1$, $n=1, 2, \cdots$. The solution ϕ (ϕ a linear functional over B) of the system $\phi(x_i)=a_i$, $i=1, 2, \cdots$, where the a_i are preassigned numbers, is investigated. It is shown that if solutions ϕ exist there is precisely one for which $\|\phi\|$ is a minimum in case B has a regular norm. If B has a uniformly regular norm and if ϕ_n represents the minimal solution of the partial system

$\phi(x_i)=a_i$, $i=1, 2, \cdots, n$, then the original system has a solution if and only if $\lim_{n\to\infty}\|\phi_n\|<\infty$, and then the minimal solution ϕ satisfies $\phi=\lim_{n\to\infty}\phi_n$. *E. R. Lorch.*

Hyers, D. H. A generalization of Fréchet's differential. Proc. Nat. Acad. Sci. U. S. A. **27**, 315–316 (1941). [MF 4581]

A definition of differential in a linear topological space. Various properties and comparisons are given without proof.
F. J. Murray (New York, N. Y.).

Michal, A. D. Higher order differentials of functions with arguments and values in topological abelian groups. Revista Ci., Lima **43**, 155–176 (1941). [MF 4846]

Here the nth differential is defined in terms of a limit involving the difference of the function and the first $n-1$ differentials. So defined the nth order differential is unique and the usual existence theorem for the differential of a linear combination is given. The second and third order differential of a function of a function are shown to exist under the usual hypotheses; the result for higher values of n is still unobtained. For Banach spaces the definition is shown to be equivalent to that of a more specialized type of nth order differential. The latter is obtainable as a limit involving just the values of the function itself.
F. J. Murray (New York, N. Y.).

Van der Lijn, Gaston. Les polynomes abstraits. (Suite et fin) Bull. Sci. Math. (2) **64**, 163–196 (1940). [MF 5232]

[The previous communications appeared in the same Bull. 64, 55–80, 102–112; cf. these Rev. 1, 259; 2, 222, 419.] Let L denote the set of μ-summable functions on a set S. Let $U(x, y)$, x and y in L, denote a continuous bilinear transformation with values in a Banach space X. The set $S\times S$ of pairs $\{P, Q\}$, P and Q in S, has a measure $\bar{\omega}$ in it determined by the measure μ in S. Given U then, there is an additive function $\varphi(E)$ of the $\bar{\omega}$-measurable sets of $S\times S$, with values in X, such that, if E is $\bar{\omega}$-measurable, $|\varphi(E)|\leqq C\bar{\omega}(E)$ and $U(x, y)=\int xyd\varphi(E)$. The inequality $|\varphi(E)|\leqq C\bar{\omega}(E)$, which is a consequence of the norm of L, insures that $\varphi(E)$ is completely additive. This result generalizes to the case in which x and y belong to different spaces of the type of L and also to the case where U is multilinear. Since monomials on abstract spaces are determined by symmetric multilinear U's, the discussion has application in the theory of abstract polynomials.
F. J. Murray (New York, N. Y.).

Halmos, Paul R. Statistics, set functions, and spectra. Rec. Math. [Mat. Sbornik] N.S. **9**(51), 241–248 (1941). (English. Russian summary) [MF 4551]

The author proposes a definition of statistical independence of elements belonging to a certain partially ordered abstract ring and shows that theorems analogous to classical theorems in probability (Tchebycheff's inequality, law of large numbers, etc.) can be obtained.
M. Kac (Ithaca, N. Y.).

Halmos, Paul R. The decomposition of measures. Duke Math. J. **8**, 386–392 (1941).

If S_t is a surface in three dimensions, and if as t varies S_t sweeps out a volume Ω, volume in Ω can be evaluated by defining an area on each S_t and integrating over t. The author generalizes this concept to abstract spaces, showing that, under suitable hypotheses, an abstract space Ω on which a measure is defined can be expressed as a sum of such spaces Ω_t, measure in Ω becoming a certain integral over t of Ω_t-measures. He uses this theorem to prove that, if T is a measure preserving transformation acting on the points of a space Ω, Ω can be split into invariant subspaces Ω_t in each of which a measure is defined, such that T is metrically transitive on Ω_t, and that Ω-measure is the integral over t of the Ω_t-measures. The hypotheses imposed on Ω are not of a topological nature. The corresponding theorem for a family of transformations depending on a continuous parameter (involving topological considerations) is due to von Neumann [Ann. of Math. (2) 33, 601–618 (1932)]. *J. L. Doob* (Urbana, Ill.).

Dieudonné, Jean. Sur le théorème de Lebesgue-Nikodym. Ann. of Math. (2) **42**, 547–555 (1941). [MF 4301]

If $x, y, \cdots$ are elements of a vector-ring-lattice L, if $J(x)$ is a positive linear functional with the properties of a norm, if another norm $\|x\|$ such that $|J(xy)|\leqq\|x\|\cdot|J(y)|$ exists, and if, for any linear functional $V(x)$ for which $0\leqq V\leqq J$, the relation

$$V(|x|)=\sup_{y\geqq 0,\, xy\leqq x}|V(xy)|$$

holds, then corresponding to each linear functional $V(x)$ which is absolutely continuous relative to $J(x)$, any element $x\geqq 0$ and any number $\epsilon>0$ there exists an element y in L, such that for any partitioning $x=x_1+\cdots+x_k$, $x_i\geqq 0$, the inequality $\sum_{i=1}^{k}|V(x_i)-V(x_iy)|\leqq\epsilon$ holds. The functional V is absolutely continuous relative to the functional J if $V=\sup_{n\geq 1}\inf(V, nJ)$. [See F. Riesz, Ann. of Math. (2) 41, 174–206 (1940); these Rev. 1, 147; and S. Bochner and R. S. Phillips, Ann. of Math. (2) 42, 316–324 (1941); cf. these Rev. 2, 315.] The following well-known theorem is a special case. If $V(t)$ is absolutely continuous and $x(t)\geqq 0$ is continuous in $0\leqq t\leqq 1$ and $\epsilon>0$, then there exists a continuous $y(t)$ such that

$$\int_0^1 x(t)d_t\left|V(t)-\int_0^t y(t)dt\right|\leqq\epsilon.$$

S. Bochner (Princeton, N. J.).

***Murray, F. J. An Introduction to Linear Transformations in Hilbert Space.** Annals of Mathematics Studies, no. 4. Princeton University Press, Princeton, N. J., 1941. ix+135 pp. $1.75.

This excellent little book furnishes a rapid and self-contained introduction to the theory of symmetric transformations in Hilbert space. Although other books on the subject are already at hand, they are out of the reach of many beginners in this field on account either of the richness of detail of the treatment or of the demands made upon the reader's knowledge of other subjects. The present treatment is so elementary as to be within the grasp of a good graduate student. This is a commendable achievement. It should be pointed out that the subject with which we are concerned is not, as indicated in the title, the structure of general linear transformations in Hilbert space but rather, in particular, of symmetric transformations and those other transformations which can be constructed from them (their functions), such as the unitary and normal transformations. The general linear transformation, whose structure is very much more complicated, is far from having yielded all its secrets.

The author first gives the fundamentals of the structure of the abstract space (manifolds, orthogonality, etc.) and adduces the usual examples of such spaces. Then follows a discussion of transformations including the basic notions of closure, boundedness, adjoint, symmetry. Next, a treatment of the weak topology with some fundamental results involving the bounded character of closed transformations defined over the entire space. The development of these two chapters seems very satisfactory. At this point the author strikes out to establish the integral form of self-adjoint transformations (s.a.tr.) and unitary transformations. This necessitates a discussion of the calculus of projections and of the Riemann-Stieltjes integral. A great charm of this theory is that the goal may be attained by such a variety of methods. In fact, it is almost de rigueur for each new treatment to be different from its predecessors. The author treats the bounded case first by establishing the homomorphism between continuous functions of a real variable and of a s.a.tr. H. For an an unbounded H he proceeds as follows: He proves that, for an arbitrary T, $(1+T^*T)^{-1}$ is s.a. and bounded, hence T^*T has an integral representation. Next he expresses T as a product of a definite s.a.tr. with a partially isometric transformation. The latter then gives the two manifolds in which H is respectively positive and negative definite. The book terminates with a discussion of symmetric transformations which are not s.a. and with a brief indication of further developments and applications. *E. R. Lorch.*

Frola, Eugenio. Sulle trasformazioni involutorie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 21–28 (1941). [MF 4792]

The author considers a real vector space S in which there is a "norm" $\|x\|$ such that $\|ax\|=|a|^r\|x\|$ for every real number a and some real r; he supposes that the space possesses a generalized inner product $|x_1, x_2|$ such that $|x_1, x_2|=|x_2, x_1|$, $|x, x|=\|x\|$, and $|x_1, x_2|=\|x_1\|=\|x_2\|$ implies $x_1=x_2$. In particular, a real normed linear space satisfies his hypotheses with $|x_1, x_2|=\frac{1}{2}\{\|x_1+x_2\|-\|x_1-x_2\|\}$. A transformation T of S into itself is called autoreciprocal if there is a generalized inner product such that $|x_1, T(x_2)|=|x_2, T(x_1)|$ for all x_1 and x_2; T is involutory if $T[T(x)]=x$; T is isonormal if $\|T(x)\|=\|x\|$. Theorem 1: An autoreciprocal transformation is involutory if and only if it is isonormal. Theorem 2: A distributive T is involutory if and only if there are two linear subspaces S_1 and S_{-1} of S having only the zero element in common, such that every element of S is a linear combination of elements of S_1 and S_{-1}, and such that S_1 is invariant under T, while if $x\epsilon S_{-1}$, $T(x)=-x$. *R. P. Boas, Jr.* (Durham, N. C.).

Julia, Gaston. Sur une définition d'opérateurs linéaires dans l'espace hilbertien. C. R. Acad. Sci. Paris **212**, 733–736 (1941). [MF 4929]

Let H be the Hilbert space of all sequences $\{x_i\}$ with $\sum|x_i|^2<\infty$ and let $e_1, e_2, \cdots$ be a complete orthonormal set in H. The author considers the problem of finding a linear operator A on a domain D to H such that $A_n=Ae_n$, where $A_1, A_2, \cdots$ is a given set of elements in H. Define D as the linear set of all $x=\{x_i\}$ in H such that $\sum x_iA_i$ converges and define Ax to be $\sum x_iA_i$. It is shown that, if $D=H$, then A is a bounded operator and its adjoint is $Bx=\sum(x, A_n)e_n$ for all x such that $\sum(x, A_n)e_n$ is in H. The proof follows from the fact that B is defined on H and hence is bounded. Conversely, if the operator B is defined for all x in H, then $D=H$ and A is bounded. The operator A is Hermitian if and only if $(e_m, A_n)=\overline{(e_n, A_m)}$. Finally the author finds a condition intrinsic to the set (A_n) that the operator A be bounded. There are several misprints, which might be confusing. *H. H. Goldstine.*

Nakano, Hidegorô. Über den Beweis des Stoneschen Satzes. Ann. of Math. (2) **42**, 665–667 (1941). [MF 4965]

A new proof of the theorem due to Stone that a one parameter group of rotations U_t in Hilbert space is generated by a self-adjoint transformation H, $U_t=\exp(itH)$. The procedure is as follows: Let $A=\int_0^\infty e^{-t}U_tdt$; then $A+A^*=2AA^*$ and $Af=0$ implies $f=0$. Thus $W=-2A+1$ is unitary and is the Cayley transform of a self-adjoint H. The latter is the transformation desired in the theorem. *E. R. Lorch* (New York, N. Y.).

Nakano, Hidegorô. Unitärinvariante hypermaximale normale Operatoren. Ann. of Math. (2) **42**, 657–664 (1941). [MF 4964]

The author gives conditions for the unitary equivalence of two hypermaximal normal operators N_1 and N_2 in Hilbert space: $N_1=U^*N_2U$. This is done in terms of rings of projections generated by the resolution of the identity $E(Z)$ of a given N. The notion of dimension of a ring is introduced and the complex plane is subdivided into mutually exclusive sets $Z_\infty, Z_1, Z_2, \cdots$ such that the ring associated with $E(Z_i)$ is of dimension i. In order that N_1 and N_2 be equivalent it is necessary and sufficient that the sets Z_i and that the null sets of $E(Z)$ formed for both operators be identical. *E. R. Lorch* (New York, N. Y.).

Eidelheit, M. On isomorphisms of rings of linear operators. Studia Math. **9**, 97–105 (1940). (English. Ukrainian summary) [MF 5258]

For a real Banach space E let $R(E)$ be the set of all bounded linear operators $y=U(x)$ with domain E and range in E. With customary definitions of the operations $R(E)$ is a linear ring with unit element, and can be considered a Banach space when normed by (1) $\|U\|=\sup_{\|x\|\leq 1}|U(x)|$. This norm satisfies the relation (2) $\|U_1U_2\|\leqq\|U_1\|\cdot\|U_2\|$. The main theorems proved in this memoir are the following: Th. 1. Any norm in $R(E)$ which makes it complete and satisfies (2) is equivalent to (1). Th. 2. Two rings $R(E_1)$ and $R(E_2)$ are algebraically isomorphic if and only if the corresponding spaces E_1 and E_2 are isomorphic. If $V=\Phi(U)$ is an isomorphism between $R(E_1)$ and $R(E_2)$, there exists an isomorphism $A(x)=y$ between E_1 and E_2 such that $\Phi(U)=AUA^{-1}$. Th. 3. Let $V=\Phi(U)$ be a one-to-one continuous and multiplicative transformation of $R(E_1)$ into $R(E_2)$, E_1 being at least two dimensional. Then $\Phi(U)$ is also additive, and hence of the form AUA^{-1}. Theorem 3 is false if E_1 is one dimensional. *J. A. Clarkson.*

Gelfand, I. Normierte Ringe. Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941). (German. Russian summary) [MF 4486]

Gelfand, I. Über absolut konvergente trigonometrische Reihen und Integrale. Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 51–66 (1941). (German. Russian summary) [MF 4490]

If a commutative ring R has a unit element e, the unit element is absent from each (proper) ideal and hence every ideal is contained in some maximal ideal. In particular it is easy to conclude that an element x of R has an inverse x^{-1} if and only if it is contained in no maximal ideal M of R. Now, if R is a normed commutative ring, that is, a Banach space with complex scalars and with a continuous product

xy such that $\|xy\| \leqq \|x\| \cdot \|y\|$, then for all maximal ideals the quotient rings R/M are isomorphic with one another, all being isomorphic with the normed field of complex numbers. This gives rise to a ring of complex valued functions $x(M)$ with the properties $e(M)=1$ and $|x(M)| \leqq \|x\|$; also for $M_1 \neq M_2$ there exists an $x \varepsilon R$ such that $x(M_1) \neq x(M_2)$. In particular, an element x of R has an inverse if (and only if) $x(M) \neq 0$ for all M. A case of special interest arises if R is the ring of periodic complex-valued functions $f(t) = \sum_{-\infty}^{\infty} a_n e^{int}$ with finite norm $\|x\| = \sum_{-\infty}^{\infty} |a_n|$. The author verifies that corresponding to each M there exist a point t_0 such that $x(M) = f(t_0)$, and thus he obtains a striking proof for the theorem of Norbert Wiener that whenever $f(t) \neq 0$ the function $1/f(t)$ has again a Fourier series of finite norm.

The author extends this statement in several versions to the case of a factor α_n appearing in the expression of the norm and to the case of an integral replacing the sum. For instance, if $\alpha(t)$ is continuous in $-\infty < t < \infty$ and $\alpha(t+s) \leqq \alpha(t) \cdot \alpha(s)$; if

$$\int_{-\infty}^{\infty} |f(t)| \alpha(t) dt < \infty \quad \text{so that} \quad F(u) = a + \int_{-\infty}^{\infty} f(t) e^{itu} dt$$

is convergent in the strip

$$\lim_{t\to\infty} \frac{\log |\alpha(t)|}{-t} \leqq \Im(u) \leqq \lim_{t\to\infty} \frac{\log |\alpha(-t)|}{t};$$

if $F(u) \neq 0$ in this strip and $a \neq 0$; then $1/F(u)$ can be represented in this strip in the form $b + \int_{-\infty}^{\infty} g(t) e^{itu} dt$, where again $\int_{-\infty}^{\infty} |g(t)| \alpha(t) dt < \infty$.

Wiener's theorem has been extended by P. Lévy in the following way. If $f(t)$ has finite norm and if an analytic function $\Phi(f)$ is analytic on the closure of the values of $f(t)$, then $\Phi(f(t))$ has again finite norm. The author can also handle this theorem by appropriately defining, in a general R, the relation $y = \Phi(x)$, where $\Phi(f)$ is an ordinary analytic function; he shows that, given x and $\Phi(f)$, there will exist an element $\Phi(x)$ if and only if the domain of analyticity of $\Phi(f)$ will include the closure of the points $x(M)$ for all maximal ideals M.

The author also makes the space $\mathfrak{M}$ of all maximal ideals M into a bicompact Hausdorff space by defining neighborhoods of M_0 in the customary way by inequalities $\{|x_i(M) - x_i(M_0)| < \epsilon\}$, $i = 1, \cdots, n$, n arbitrary. The family of functions $\{x(M)\}$ is included and everywhere dense (for uniform convergence) in the space $C(\mathfrak{M})$ of all continuous functions on $\mathfrak{M}$. In order that every element of $C(\mathfrak{M})$ be a function $x(M)$ it is necessary and sufficient that $x(M) = 0$ for all M shall imply $x = 0$. In this connection the author shows that $\max |x(M)|$, $M \varepsilon \mathfrak{M}$, equals $\lim_{n\to\infty} (\|x^n\|)^{1/n}$. Therefore if we define the generalized radical of R to consist of those elements for which $\lim (\|x^n\|)^{1/n} = 0$ (that is, of generalized nilpotent elements), then the generalized radical is the intersection of all maximal ideals. *S. Bochner.*

Gelfand, I. Ideale und primäre Ideale in normierten Ringen. Rec. Math. [Mat. Sbornik] N.S. 9(51), 41–48 (1941). (German. Russian summary) [MF 4488]

A closed ideal I of a normed commutative ring [see the preceding review] is called primary if it is contained in only one maximal ideal. It is primary if for every element x in R/I either the inverse exists or $(\|x^n\|)^{1/n} \to 0$. The author also discusses the existence of primary ideals which are contained in a given maximal ideal M_0. For instance, if R has a generating element x, if $x(M)$ is real, and if, for all complex numbers λ in the neighborhood of $\lambda_0 = x(M_0)$,

$$\|(x - \lambda e)^{-1}\| = o(1/|\Im(\lambda - \lambda_0)^n|),$$

where n is a positive integer, then only the smallest closed ideals containing any one of the elements $(x - \lambda_0 e)^k$, $k = 1, \cdots, n-1$, may be primary ideals in M_0. Other results refer to rings R in which every closed ideal is an intersection of maximal ones. *S. Bochner* (Princeton, N. J.).

Gelfand, I. und Šilov, G. Über verschiedene Methoden der Einführung der Topologie in die Menge der maximalen Ideale eines normierten Ringes. Rec. Math. [Mat. Sbornik] N.S. 9(51), 25–39 (1941). (German. Russian summary) [MF 4487]

The paper discusses two ways of topologizing the space $\mathfrak{M}$ of maximal ideals M in a commutative normed ring R. The first topology in which neighborhoods are given by systems of relations $|x_i(M) - x_i(M_0)| < \epsilon$, $i = 1, \cdots, n$, was introduced and discussed by Gelfand [cf. the second review above] and the authors give a new analysis, of a more algebraic nature, in the special case in which every element x of R possesses a conjugate element $\bar{x}$ (with $\bar{x}(M) = \overline{x(M)}$). The second topology is purely algebraical and similar to the topology of Stone and Wallman: the closure of a point set A in $\mathfrak{M}$ includes every element M_0 of $\mathfrak{M}$, which as a point set of R includes the intersection of the sets M belonging to A. The topologies may differ. An interesting case in which they are equivalent will arise if our ring R is the ring of all continuous functions on a completely regular Hausdorff space S. The space $\mathfrak{M}$ in either topology is the bicompact closure of S, and each continuous function $x(t)$ on S has a continuous extension onto all of $\mathfrak{M}$. *S. Bochner.*

Ambrose, Warren. Representation of ergodic flows. Ann. of Math. (2) 42, 723–739 (1941). [MF 4970]

A space Ω is a measure space if a completely additive measure is defined on Ω such that $0 < m\Omega < \infty$, such that there exists a measurable subset M for which $0 < mM < m\Omega$, and such that the measure is completed in the sense that any subset of a set of measure zero is measurable. Let T be a 1-1 measure-preserving transformation of the measure space Ω onto Ω; let $\Omega \times L$ denote the product space of Ω and the real axis L, where measure is defined multiplicatively in terms of the given measure on Ω and Lebesgue measure on L; let $f(P)$ be a real-valued integrable function defined on Ω with $f(P) > c > 0$; and let $\bar{\Omega}$ be the set of points in $\Omega \times L$ defined by (P, x), $P \varepsilon \Omega$, $0 \leqq x < f(P)$. Let the transformation T_t of $\bar{\Omega}$ onto $\bar{\Omega}$ be defined as follows:

$$T_t(P, x) = (T^n P, t + x - f(P) - \cdots - f(T^{n-1}P)),$$
$$n > 0; \quad 0 \leqq t + x - f(P) - \cdots - f(T^{n-1}P) < f(T^n P);$$
$$T_t(P, x) = (P, t + x), \qquad 0 \leqq t + x < f(P);$$
$$T_t(P, x) = (T^{-n}(P), \quad t + x + f(T^{-1}P) + \cdots + f(T^{-n}P)),$$
$$n > 0; \quad -f(T^{-n}P) \leqq t + x + f(T^{-1}P) + \cdots + f(T^{-n+1}P) < 0.$$

Then $\bar{\Omega}$ is a measure space and T_t is a flow (a one-parameter family of 1-1 measure-preserving transformations of a measure space onto itself satisfying the group property $T_t T_s = T_{t+s}$) in $\bar{\Omega}$. This flow is termed the flow built on T under $f(P)$. [Particular forms of this type of flow have been constructed by von Neumann [Ann. of Math. (2) 33, 587–642 (1932)] to realize some of the possibilities in the spectra of flows.] The flow T_t on Ω' is isomorphic to the flow S_t on Ω'' if there exists a measure-preserving transformation R (defined to within invariant sets of measure zero) of Ω' onto Ω'' such that $T_t = R^{-1} S_t R$. The principal result of the present

paper is a proof that every measurable ergodic flow is isomorphic to a flow built under a function. With the aid of this representation the author is able to derive several other interesting results. Except in two highly degenerate cases, the orbit of any point of any measurable flow is a measurable set of measure zero. A measurable ergodic flow is isomorphic to a flow built under a constant function if and only if the corresponding group U_t of unitary transformations in Hilbert space has a characteristic function with corresponding characteristic value not zero. Extending a theorem of von Neumann [loc. cit.] for differentiable flows, the author proves that for any measurable ergodic flow the band spectrum is either a null set or the whole real line. *G. A. Hedlund* (Charlottesville, Va.).

Ambrose, Warren. Change of velocities in a continuous ergodic flow. Duke Math. J. 8, 425–440 (1941).

If T_t is a continuous ergodic flow on a separable metric space of finite measure, then, by a change of velocity along each trajectory, it is possible to obtain a flow S_t with a point spectrum with respect to a new measure which is equivalent to the original measure and is invariant under S_t. (S_t is not always continuous.) The author proves this result by showing that S_t is isomorphic to the flow $T_t(P, x)$ (defined on a measure space $\Omega\times I$ with the ordinary product measure, where $\Omega=\{P\}$ is a measure space and $I=\{x\}$ is the interval of real numbers: $0\leqq x<1$) of the following form: $T_t(P, x) =(T^n(P),\ x+t-n)$, for $n-x\leqq t<n+1-x$, $n=0, \pm1, \pm2, \cdots$, where T is a measure preserving transformation defined on Ω. Also, S_t (or equivalently $T_t(P, x)$) has a pure point spectrum if and only if T has a pure point spectrum. Meanwhile, more general results were obtained by the author [cf. the preceding review]. *S. Kakutani.*

Calculus of Variations

Cinquini, Silvio. Il calcolo delle variazioni. Period. Mat. (4) 20, 205–217, 269–288 (1940). [MF 5374]

Mayer, Walther. Calculus of variations. Uspekhi Matem. Nauk 9, 254–312 (1941). (Russian) [MF 5079]

This is an expository paper based on the author's Princeton lectures.

Hestenes, Magnus R. An analogue of Green's theorem in the calculus of variations. Duke Math. J. 8, 300–311 (1941).

The author discusses necessary and sufficient conditions for

$$(1)\qquad \iint_A (uz_x+vz_y+wz)dxdy=\int_C z(udy-vdx)$$

to hold for all functions z of class C' on $A+C$, C being the boundary of A. The fundamental assumptions are that C consists of a finite number of simple, closed, rectifiable curves, that u and v are continuous on $A+C$, and that w is summable over A. The author's first result is that (1) holds for all admissible z if and only if it holds for all such z which vanish on and near C. The second is that (1) holds for all admissible z if and only if $\iint_R wdxdy=\int_{R^*} udy-vdx$ for all rectangles R whose closures are in A (R^*=boundary of R). If, on each cell R, u is absolutely continuous (AC) in x for almost all y and v is AC in y for almost all x with u_x and v_y summable over A, then the second result is equivalent to the condition that $w=u_x+v_y$ almost everywhere. By putting $z\equiv1$ in (1), the author obtains Green's formula: $\iint_A(u_x+v_y)dxdy=\int_C udy-vdx$ under the above general assumptions on u, v, A and C. To prove the above results the author gives an elementary proof of the fact that such regions A may be approximated from the interior by regions A_n which are bounded by polygons in such a way that the bounding curves of the A_n have uniformly bounded lengths and tend to those of A in the sense of Fréchet. This last result is used to give a simple proof of Cauchy's theorem for regions A of the above type. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Ermilin, K. Sur l'extrémum des intégrales des fonctions discontinues. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 269–276 (1941). (Russian. French summary) [MF 5178]

The author considers the problem of minimizing an integral $\int F(x, y, x', y')dt$ in which the integrand is discontinuous along the positive y'-axis; at each point on this axis its value is the mean of its left and right limits. The curves permitted are of class D' and contain a finite number of vertical segments. The notion of "nearness" of two curves adds to the usual concept the requirement that they have an equal number of vertical segments, the ends of the nth segment of the one curve lying near those of the nth segment of the other curve. The author announces necessary conditions and sufficient conditions for a relative minimum, including special conditions for the end-points of vertical segments. A distinctive feature is that the necessary condition of Weierstrass cannot be satisfied unless the tangent vectors (x', y') lie outside of a certain sector

$$a\leqq x'[x'^2+y'^2]^{-\frac{1}{2}}\leqq 0, \qquad y'>0.$$

This does not hinder the application to the discontinuous solutions considered by Razmadze [Math. Ann. 94 (1925)]. An example is presented in detail. *E. J. McShane.*

Damköhler, Wilhelm. Zur Frage der Äquivalenz indefiniter Variationsprobleme mit definiten. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1940, 1–14 (1940). [MF 4573]

The author outlines a proof of the following theorem: Let $F(x, x')$ be the integrand of a variation-problem which is positive regular in a bounded closed region B of x-space. Let $L(C)$ denote the length of a curve C, and let $q(C) =\int_C F(x, x')dt/L(C)$ have the positive lower bound m in the class of all closed rectifiable curves C in B. Then to every $\epsilon>0$ there corresponds a function $S(x)$ such that $F(x, x') +S_{x_i}x_i'\geqq m-\epsilon$ for all x in B and all x' such that $\sum x_i'^2=1$. Certain modifications are indicated relating to the application of the method to problems of Lagrange. *L. M. Graves* (Chicago, Ill.).

McShane, E. J. On the second variation in certain anormal problems of the calculus of variations. Amer. J. Math. 63, 516–530 (1941). [MF 4675]

The author proves that, if $y_i=y_i(x)$, $x_1\leqq x\leqq x_2$, affords a strong relative minimum for a problem of Lagrange with variable end points, and the order of anormality of this arc does not exceed one, then there exists a set of multipliers $\lambda_0\geqq0$, $\lambda_1(x)$, $\cdots$, $\lambda_m(x)$, $[\lambda_0+\sum_\alpha|\lambda_\alpha(x)|\neq0]$, such that with these multipliers the arc satisfies the Du Bois-Reymond equations and transversality condition, the necessary con-

ditions of Weierstrass and Clebsch and, moreover, the second variation is non-negative for all admissible variations. The present paper complements a previous paper of the author [Amer. J. Math. **61**, 809–819 (1939); these Rev. **1**, 78] in which the above result, except for the non-negative character of the second variation, was proved without restriction on the order of anormality of the minimizing arc. In conclusion, an example is given to show that the restriction on the order of anormality is essential for the validity of the conclusion of the present paper.

W. T. Reid (Chicago, Ill.).

Kantorovitch, L. On the convergence of variational processes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 107–111 (1941). [MF 4262]

Consider the problem of minimizing

$$I[u]=\iint_D [a(\partial u/\partial x)^2+b(\partial u/\partial y)^2+cu^2+2fu]dxdy,$$

where a, b, c, f are functions of x, y and $a, b>0$, $c\geqq 0$, among all functions defined over the domain D and vanishing on the boundary of D. Let u be a solution of the problem satisfying $\int_\alpha^\beta(\partial u/\partial y)^2dy\leqq K$ for every line segment parallel to the y-axis and contained in D. Let u_n be a minimizing sequence, so that $\epsilon_n=I[u_n]-I[u]\to 0$. The author proves by simple estimations involving the Dirichlet integral that u_n will converge uniformly to u if

$$\int_\alpha^\beta (\partial u_n/\partial y)^2dy\leqq K_n$$

and $\epsilon_n \log K_n\to 0$. This theorem is applied to the case when u_n is a polynomial or trigonometric polynomial in y, by estimating $\int_\alpha^\beta(\partial u_n/\partial y)^2dy$. The result is that u_n converges uniformly to u if $\epsilon_n \log n\to 0$, and then

$$|u_n-u|=O((\epsilon_n \log (n/\epsilon_n))^{\frac{1}{2}}).$$

M. Shiffman (New York, N. Y.).

Kantorovitch, L. On the convergence of the method of reduction to ordinary differential equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 585–588 (1941). [MF 4465]

The author shows how his method of reduction to ordinary differential equations [Bull. Acad. Sci. URSS (7) **1933**, 647–652 (1933)] will yield a minimizing sequence converging uniformly to the solution of the minimum problem: $I(u)$ = minimum among functions vanishing on the boundary [see the preceding review]. Suppose that the domain D is bounded by the lines $x=0$, $x=1$, $y=g(x)$, $y=h(x)$, where $g(x)$, $h(x)$ have bounded first derivatives and $h(x)>g(x)$. And suppose that the solution $u(x, y)$ has derivatives u_{yy}, u_{xy} which are integrable together with their squares in D. It is shown, for example, that functions $\bar{u}_n(x, y)$ can be found which have the form

$$\bar{u}_n(x, y)=\sum_{k=1}^{n} f_k(x) \sin k\pi\frac{y-g(x)}{h(x)-g(x)},$$

and for which $\epsilon_n=I(u_n)-I(u)=O(n^{-2})$. The sequence u_n obtained by the method of reduction to differential equations will therefore converge uniformly to u [by the paper reviewed above] since it yields a smaller value of ϵ_n.

M. Shiffman (New York, N. Y.).

Douglas, Jesse. Solution of the inverse problem of the calculus of variations. Trans. Amer. Math. Soc. **50**, 71–128 (1941). [MF 4870]

The author solves the following problem. Given a four-parameter family of curves in (x, y, z)-space satisfying equations $y''=F(x, y, z, y', z')$, $z''=G(x, y, z, y', z')$, to determine whether these are the extremals of some variation problem $\int\varphi dx=$min., and if so to find all corresponding functions φ. The problem is reduced to the study of a differential system $\mathfrak{S}$, which in turn is replaced (if consistent) by an equivalent completely integrable system, from which the degree of arbitrariness of the solution φ can be found. The principal results have already been announced [Proc. Nat. Acad. Sci. U. S. A. **26**, 215–221 (1940); see these Rev. **1**, 244].

E. J. McShane (Charlottesville, Va.).

Courant, Richard. On the first variation of the Dirichlet-Douglas integral and on the method of gradients. Proc. Nat. Acad. Sci. U. S. A. **27**, 242–248 (1941). [MF 4435]

Let Γ be a set of k distinct Jordan curves in N-space; let B, B', etc. denote k-fold connected regions bounded by analytic curves C, C', etc.; let $\mathfrak{S}$ denote the class of vector functions $\mathfrak{x}$ (1) which are continuous on $B+C$, (2) which have piecewise continuous derivatives in B, (3) which have finite Dirichlet integrals and (4) which map the curves of C in a continuous monotone way onto those of Γ; and let $\mathfrak{P}$ be the subset of $\mathfrak{S}$ in which the $\mathfrak{x}$ are harmonic. Let λ and μ be functions continuous on $B+C$ with bounded piecewise continuous derivatives; let $u=u'+\epsilon\cdot\Lambda(u', v', \epsilon)$, $v=v'+\epsilon\cdot M(u', v', \epsilon)$, where $\Lambda(u', v', 0)=\lambda$, $M(u', v', 0)=\mu$, be a transformation of $B'+C'$ into $B+C$; let $\mathfrak{z}_\epsilon(u', v')=\mathfrak{x}(u, v)$; let $\mathfrak{h}_\epsilon(u', v')$ be the harmonic function coinciding with $\mathfrak{z}_\epsilon$ on C'; and let $V(\mathfrak{x}, \lambda, \mu)=\{d/d\epsilon D(\mathfrak{z}_\epsilon, B_\epsilon')\}_{\epsilon=0}$ and $S(\mathfrak{x}, \lambda, \mu)=\{d/d\epsilon D(\mathfrak{h}_\epsilon, B_\epsilon')\}_{\epsilon=0}$. The author shows that the variations V and S are equal if $\mathfrak{x}$ is harmonic and then is able to show the relationship between his theory of first variation of the Dirichlet-Douglas integral and that of Morse and Tompkins. The paper concludes with a procedure for constructing a minimal surface by a "method of steepest descent" starting from any harmonic vector in $\mathfrak{P}$ when Γ is one curve and B is the unit circle.

C. B. Morrey, Jr. (Berkeley, Calif.).

Courant, R. On a generalized form of Plateau's problem. Trans. Amer. Math. Soc. **50**, 40–47 (1941). [MF 4868]

In the place of the ordinary Plateau problem, called problem I, the author considers a more general problem II defined below and confines himself, for simplicity of exposition, to the case of one contour. Instead of requiring the vector functions $\mathfrak{x}(u, v)$ (defined on the unit circle B with boundary C) to be continuous on $B+C$ and map C in a monotone way on to the given contour Γ, the author merely requires that the vectors $\mathfrak{x}(u, v)$ satisfy the following: (1) $\mathfrak{x}(u, v)$ shall be continuous with piecewise continuous first derivatives on B; (2) all the limiting values of $\mathfrak{x}(u, v)$ as $(u, v)\to C$ lie on Γ; (3) for each $\epsilon>0$, there exists an $r_0<1$ such that the curve $x=\mathfrak{x}(r, \theta)$ (r fixed) can be continuously deformed into Γ within the ϵ neighborhood of Γ if $r>r_0$. The author proves that any solution of problem I is also a solution of problem II; that is, the minimum value of the Dirichlet integral in problem II is the same as that in problem I. The solutions of problem II always exist, if the Dirichlet integral is finite for some admissable $\mathfrak{x}$, and are minimal surfaces. To point out the significance of his result, the author constructs an example of a minimal surface which solves problem II without solving problem I.

C. B. Morrey, Jr. (Berkeley, Calif.).

Ritter, I. F. Solution of Schwarz' problem concerning minimal surfaces. Univ. Nac. Tucumán. Revista A. 1, 49–62 (1940). [MF 4054]

The Schwarz problem is the following extension of the Plateau problem: to find a minimal surface bounded by a closed chain of Jordan arcs alternating with planes, so that the minimal surface meets the plane orthogonally. On the basis of Courant's procedure in the Plateau problem and a principle of reflection across planes, a sufficient condition for the existence of such a minimal surface is obtained. This sufficient condition asserts that the lower bound of the areas of surfaces spanned in the chain of arcs and planes is less than the lower bound of the areas of 'degenerate' surfaces, that is, surfaces which split into two pieces, one bounded by some arcs and planes and the other bounded by the remaining arcs and planes. *M. Shiffman.*

Morse, Marston and Tompkins, C. Unstable minimal surfaces of higher topological structure. Duke Math. J. 8, 350–375 (1941).

The authors consider minimal surfaces bounded by two non-intersecting simple closed curves C_1: $x=g_1(t)$ and C_2: $x=g_2(t)$ which are such that the ratio of any smaller arc to the corresponding chord is uniformly bounded; in the above representations t is proportional to arc length and $g_k(t+2\pi)\equiv g_k(t)$, $t=1, 2$. Using their previously developed general critical point theory [see M. Morse, Ann. of Math. (2) 41, 419–454 (1940); these Rev. 1, 320; and Ann. of Math. (2) 40, 443–472 (1939)] the authors prove a theorem of which the following result is a corollary: if the C_k are separated by a hyperplane and each has a convex plane projection, and if the two C_k bound a minimal surface of minimizing type, then they bound one of non-minimizing type.

To apply their general theory and overcome the difficulties caused by degenerate representations, they observe first that any continuous transformation h of the unit circle into itself may be represented as a product of the form φT, where T is a Möbius transformation and φ is a restricted transformation, that is, is continuous and leaves 3 fixed points invariant. Let $x=p_1(\alpha)$ and $x=p_2(\alpha)$ be representations of C_1 and C_2; then $D(p_1, p_2, \rho)$ denotes the Dirichlet integral of the harmonic vector defined on the circular ring $0\leqq\alpha\leqq 2\pi$, $\sigma_1\leqq r\leqq\sigma_2$ with $\rho=\sigma_1/\sigma_2$. The authors define a space Π of points P: $(\varphi_1, \varphi_2, T_1, T_2, \rho)$, where φ_1 and φ_2 are restricted and T_1 and T_2 are Möbius transformations or are degenerate; if $\rho=0$, points having the same (φ_1, φ_2) are considered identical, otherwise there are no identifications. On Π, the authors define the function

$$W(P)=D(p_1)+D(p_2)+\Omega(q_1, q_2, \rho),\quad P=(\varphi_1, \varphi_2, T_1, T_2, \rho),$$
$$p_i(\alpha)=g_i[\varphi_i(\alpha)],\quad q_i(\alpha)=g_i[\varphi_i\{T_i(\alpha)\}],\quad i=1, 2,$$

where $\Omega(q_1, q_2, \rho)$ is defined by the identity

$$D(q_1, q_2, \rho)=D(q_1)+D(q_2)+\Omega(q_1, q_2, \rho)$$

and $D(h)$ denotes the Dirichlet integral of the harmonic vector determined by the vector h on the boundary of any circle. The authors are able to metrize Π and to show that $W(P)$ is regular at infinity, that the set where $W\leqq c$ is compact, that W is weakly upper reducible at each point where it is finite, and that homotopic critical points of W give rise to minimal surfaces (via the representations q_i). Also $W(P)\geqq D(q_1, q_2, \rho)$, and degenerate representations q_i do not introduce the trivial minima as they do for the functional D. *C. B. Morrey, Jr.* (Berkeley, Calif.).

TOPOLOGY

***Franklin, Philip. The four color problem.** Galois Lectures, Scripta Mathematica Library, no. 5, pp. 49–85. New York, 1941.

In this purely expository paper is given a very comprehensive introduction to the four color problem and its generalizations. Practically all of the methods and results at present in the literature are touched upon to some extent. A few (particularly those pertaining to various classical results of Heawood) are given in considerable detail. *D. C. Lewis* (Durham, N. H.).

***Bourbaki, N. Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre I. Théorie des ensembles (Fascicule de résultats).** Actual. Sci. Ind., no. 846. Hermann & Cie., Paris, 1939. viii+51 pp.

Bourbaki is a pen name of a group of younger French mathematicians who set out to publish an encyclopedic work covering most of modern mathematics. This issue is devoted to set theory and is only a digest of the proper volume. The purpose is to give the reader interested in one of the further volumes the necessary set theoretic preparation without bothering with a rigorous axiomatic approach and proofs; actually the material is arranged so excellently that most of the proofs can be easily completed. The table of contents: 1. Elements and parts of a set; 2. Functions; 3. Products of several sets; 4. Union, intersection and products of a family of sets; 5. Equivalence relations, quotient sets; 6. Ordered sets; 7. Powers, countable sets; 8. Ladders of sets and structures. The last section outlines an interesting method of treating structures, such as order, topology, group, ring, etc., on a general basis and having concepts like isomorphism defined quite generally. The method of partially ordered sets is strongly emphasized and the importance of Zorn's lemma is stressed. *S. Eilenberg.*

***Bourbaki, N. Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre III. Topologie générale. Chapitres I et II.** Actual. Sci. Ind., no. 858. Hermann & Cie., Paris, 1940. viii+132+II pp.

The first chapter entitled "Topological Structures" is devoted to the study of topological and Hausdorff spaces. The discussion is based on the concept of a filter. A non-empty family F of subsets of set X is called a filter if (1) every set containing a set of F is in F; (2) the intersection of two sets in F is in F; (3) the empty set is not in F. The filter F converges to x if every neighborhood of x contains a set of F. Using this concept of convergence a complete equivalence between neighborhood, open set and convergence topology is achieved. Other topics discussed in the chapter are continuity of transformations, products, compactness (meaning bicompactness) and connectedness.

Chapter two is devoted to uniform structures which are the modern substitute for metric spaces. With the use of filters an exceedingly elegant treatment is presented. The main results are: (1) every uniform space can be imbedded into a complete uniform space; (2) every compact space is homeomorphic with a uniform space.

Both chapters are followed by historical notes and contain many exercises of varied difficulty. The notations and terminology are rigorous and thoroughly consistent.
S. Eilenberg (Ann Arbor, Mich.).

Wilcox, L. R. A topology for semi-modular lattices. Duke Math. J. 8, 273–285 (1941).

It is shown how any topology defined on the points of an atomistic semi-modular lattice may be extended to the elements of any dimension. Under suitable hypotheses fulfilled by the subspaces of real or complex affine geometry, the extended topology yields a Hausdorff space, in which the elements of any fixed dimension are open and closed subspaces. Special cases of the definitions include tangent line, tangent plane, osculating plane and so on [cf. also Haupt, Nöbeling and Pauc, J. Reine Angew. Math. 181, 193–217 (1940); cf. these Rev. 1, 169]. *G. Birkhoff.*

Lasalle, J. P. Topology based upon the concept of a pseudo-norm. Proc. Nat. Acad. Sci. U. S. A. 27, 448–451 (1941). [MF 5166]

A topology is introduced into a set T by means of a non-negative real-valued function $\|x, d\|$, called a pseudo-norm, which is defined on T and a partially ordered system D with certain conditions put upon the function. Defining $U(d)=[x; x\epsilon T, \|x, d\|\leqq 1]$, a neighborhood topology is introduced. An equivalent formulation of the topology in terms of neighborhood axioms is given, one of these turning out to be somewhat weaker than customary. Conditions for closure and openness of sets and for continuity of functions are given in terms of pseudo-norms. *J. V. Wehausen.*

Gibert, Armando et Ribeiro, Hugo. Quelques propriétés des espaces (Cf). Portugaliae Math. 2, 110–120 (1941). [MF 4482]

A space (Cf) (sometimes called a space of finite character) is a neighborhood space such that each point of the space has one neighborhood contained in the other neighborhoods of this point. A space (Cf) is said to be a Linfield or a semi-Linfield space if the "smallest" neighborhood of each point consists of a finite or any number of points, respectively. An equivalent definition of a space (Cf) is a space with completely additive closure operation. A space with completely additive closure operation such that p and q distinct points implies $(\bar{p})$ and $(\bar{q})$ are distinct is called a discrete space of Alexandroff.

Let 1 be a set and ρ a binary relation between the elements of an ordered pair (x, y) such that for $x\epsilon 1$ and $y\epsilon 1$ either $x\rho y$ or $\sim x\rho y$ and always $\sim x\rho x$ for $x\epsilon 1$. Also let $^-$ be the operation such that, for $X\subset 1$, $z\epsilon\bar{X}$ if and only if either $z\epsilon X$ or there exists an $x\epsilon X$ such that $z\rho x$. The author proves: The system $[1, ^-]$ is a space (Cf) (in which $^-$ is the closure operation) and $x\rho y$ if and only if $x\neq y$ and $x\epsilon(\bar{y})$. Furthermore, a space (Cf) is a semi-Linfield space if ρ is symmetric and is a discrete Alexandroff space if ρ is anti-symmetric and transitive. Also studied are functions continuous on a space (Cf) to a space (Cf) and topologies reciprocal in a space (Cf), where for a point the "smallest" neighborhood in the reciprocal topology is the closure of the set (x) in the given topology. *J. F. Randolph* (Ithaca, N. Y.).

Ribeiro, Hugo. Caractérisations des espaces réguliers normaux et complètement normaux au moyen de l'opération de dérivation. Portugaliae Math. 2, 13–19 (1941). [MF 4500]

Fréchet said that it would be of use to have two definitions for normal and completely normal spaces, one based directly on the choice of neighborhoods and the other on the operation of derivation. This program is here carried through. The axioms for the operation of derivation for regular spaces are (I) $(A+B)'=A'+B'$; (II) $A''\subset A'$; (III) $A'=0$ if A consists of a single point; (IV) if a point b and a set B_1 are separated and both belong to the derivative of a set A, there exists a decomposition of A into two non-empty disjoint sets A_1 and A_2 such that $A_1'(b)=0$ and $A_2'B_1=0$.

The axioms for normal spaces (or those for completely normal spaces) are obtained by merely substituting $(D_3)((D_4))$ in place of axiom (IV), where (D_3) if B_1 and B_2 belong to the derivative of a set A and $B_1\neq 0$, $B_2\neq 0$, $B_1B_2+B_1B_2'+B_2B_1'+B_1'B_2'=0$, there exists a decomposition of A into two non-empty disjoint sets A_1 and A_2 such that $A_1'B_2=0$ and $A_2'B_1=0$; (D_4) same statement as (D_3) except that $B_1B_2+B_1B_2'+B_2B_1'=0$ instead of $B_1B_2+B_1B_2'+B_2B_1'+B_1'B_2'=0$. *J. F. Randolph* (Ithaca, N. Y.).

Monteiro, António. Les ensembles fermés et les fondements de la topologie. Portugaliae Math. 2, 56–66 (1941). [MF 4502]

The most general spaces which may be characterized by the notion of neighborhoods are referred to as the spaces (V). An abstract set 1 is said to be a space (F) if to each $A\subset 1$ is associated a unique set $\bar{A}\subset 1$ and $\bar{O}=O$, $A\subset\bar{A}$, $\bar{A}+\bar{B}\subset\overline{A+B}$, $\bar{\bar{A}}=\bar{A}$. The purpose of this paper is to (1) show that there exist spaces (V) for which the topology is not determined by the knowledge of the closed sets of the space; (2) show that the most general spaces (V) for which the topology is uniquely determined by the knowledge of the family of closed sets are precisely the spaces (F); (3) show that the spaces (F) are the spaces (V) for which the closure of a set verifies the relation $\bar{\bar{A}}=\bar{A}$.
J. F. Randolph (Ithaca, N. Y.).

Tola, José, P. Operations continuous for sequences and for neighborhoods in topological spaces. Actas Acad. Ci. Lima 4, 73–75 (1941). (Spanish) [MF 5228]

Given two topological spaces in each of which are defined topologies both by the customary closure axioms and by Fréchet's limit axioms; necessary and sufficient conditions are given without proof in order that functions from one space to the other continuous in the limit topologies be continuous in the closure topologies. *J. V. Wehausen.*

Chogoshvili, George. Über Konvergenzräume. Rec. Math. [Mat. Sbornik] N.S. 9(51), 377–383 (1941). (Russian. German summary) [MF 4560]

The author characterizes in terms of directed sets ("unbounded partially ordered systems") a topological space (following the terminology of Alexandroff-Hopf) and the separation axioms T_0, T_1, T_2, T_3 and T_4. Reference is made to the points of contact with Garrett Birkhoff's paper on Moore-Smith convergence [Ann. of Math. (2) 38, 39–56 (1937)]. *J. V. Wehausen* (Columbia, Mo.).

Sebastião e Silva, J. Sur l'axiomatique des espaces de Hausdorff. Portugaliae Math. 2, 93–109 (1941). [MF 4308]

The purpose of this paper is to characterize Hausdorff spaces with the operations of frontier, orle and border (intuitively $\mathfrak{f}(A)=\bar{A}(\overline{1-A})$, $\mathfrak{O}(A)=\bar{A}(1-A)$, $\mathfrak{b}(A)=\mathfrak{O}(1-A)$). Thus a space $[1, \mathfrak{f}]$ is a Hausdorff space if $(I_\mathfrak{f})$ $AB\mathfrak{f}(AB)=AB[\mathfrak{f}(1-A)+\mathfrak{f}(1-B)]$; $(II_\mathfrak{f})$ $\mathfrak{f}[\mathfrak{f}(A)]\subset\mathfrak{f}(A)$; $(III_\mathfrak{f})$ $\mathfrak{f}(A)\subset A$ if A is empty or consists of one point; $(IV_\mathfrak{f})$ if

distinct points a_1 and a_2 both belong to the frontier of a set A, then $A=A_1+A_2$, $A_1A_2=0$ and a_1 not $\epsilon\mathfrak{f}(A_2)$, a_2 not $\epsilon\mathfrak{f}(A_2)$. A space $[1, \mathfrak{D}]$ is a Hausdorff space if ($\mathrm{I}_{\mathfrak{D}}$) $\mathfrak{D}(A+B)=(1-B)\mathfrak{D}(A)+(1-A)\mathfrak{D}(B)$ and ($\mathrm{II}_{\mathfrak{D}}$), ($\mathrm{III}_{\mathfrak{D}}$), ($\mathrm{IV}_{\mathfrak{D}}$) are obtained from the respective axiom above by replacing frontier by orle. Similar axioms for border are also given.

J. F. Randolph (Ithaca, N. Y.).

Wallace, A. D. Separation spaces. Ann. of Math. (2) **42**, 687–697 (1941). [MF 4968]

The paper gives a set of axioms sufficient to construct a theory of connectivity of sets in terms of a new undefined concept $X|Y$, which may be read "X is separated from Y". The axioms are: (1) $0|X$ for every X, that is, the null set is separated from every set. (2) $X|Y$ implies $Y|X$. (3) $X|Y$ implies $X\cdot Y=0$. (4) $X|Y$ and $X_1\subset X$ implies $X_1|Y$. (5) $X_1|Y$ and $X_2|Y$ imply $(X_1+X_2)|Y$. On the basis of this undefined binary relation two notions of the closure of a set are defined and the usual theorems are proved for both types. These are kX, the set of all points y such that $y|X$ is false, and hX, the product of all sets Y such that $Y=kY\supset X$. (6) $y|X$ implies $y|kX$. (7) $x\neq y$ implies $x|y$. (8) If $x\epsilon X$ and $y\epsilon Y$ implies $x|Y$ and $y|X$, then $X|Y$. It is shown that axioms (1)–(7) give $kX=hX$. In a space satisfying axioms (1)–(8), the operator k satisfies the three Kuratowski closure axioms and $X|Y$ is equivalent to $X\cdot kY+Y\cdot kX=0$, that is, Lennes separation with the k-operator as closure. Conversely, if the k-operator satisfies the three Kuratowski axioms and $X|Y$ is defined by $X\cdot kY+Y\cdot kX=0$, then axioms (1)–(8) are satisfied. In an s-space (axioms (1)–(5)) the term s-connected is defined in the usual way and the classical theorems on connected sets are found to be true. Let S be a space satisfying the three Kuratowski axioms and let G be any family of closed sets of S. Then $X|Y$ may be defined as true if either set is null or if G contains a set Z such that $S-Z=U+V$, $X\subset U$, $Y\subset V$, $\bar{U}V+U\bar{V}=0$. Then axioms (1)–(4) are always true and various topologies may be obtained by varying the family G. Finally the Eilenberg-Whyburn theorem that any continuous transformation is the product of a monotonic and a light transformation is proved for s-spaces. *W. L. Ayres.*

Wallace, A. D. The acyclic elements of a Peano space. Bull. Amer. Math. Soc. **47**, 778–780 (1941). [MF 5487]

This note considers a compact locally connected continuum S, and denotes by $Q(S)$ the set of all cut-points and end-points of S. The components of $Q(S)$ are called acyclic elements of S. Using the ideas developed by Kuratowski and Whyburn [Fund. Math. **16**, 305–331 (1930)], the author characterizes acyclic elements by means of conjugate points, and shows that many of their properties may be considered as the duals of the strictly analogous properties of cyclic elements. *D. W. Hall* (Providence, R. I.).

Wallace, A. D. A fixed-point theorem for trees. Bull. Amer. Math. Soc. **47**, 757–760 (1941). [MF 5482]

This paper considers a compact (=bicompact) Hausdorff space T satisfying the following conditions: (a) T is locally connected in the sense that if U is a finite open covering of T then there is a finite open covering B contained in U and having connected sets as vertices; (b) if U is any finite open covering of T then there is a finite open covering B contained in U such that the nerve of B is a combinatorial tree. Spaces satisfying these conditions are called trees. The author has shown elsewhere [see Duke Math. J. **6**, 31–37 (1940); these Rev. **1**, 222] that an acyclic continuous curve in the usual sense is a tree in this terminology. It is shown that the product of any two subcontinua of T is itself a subcontinuum of T. The author considers mappings q which assign to each point t of a topological space a set qt in the same or another topological space. Such a mapping is said to be continuous provided that for any point t and any neighborhood U of qt there exists an open set V containing t such that for any point t' in V the set qt' is contained in U. The principal theorem of the paper states that, if q is any continuous point to continuum mapping of a tree T into itself, then there is a point t_0 in T such that t_0 lies in qt_0. The Scherrer fixed-point theorem is an immediate corollary. The author deduces the following additional consequences of his theorem: (1) No continuum admits a free monotone transformation onto a tree. (2) A monotone transformation $fM=T$ of a continuum onto a tree admits a coincidence with any continuous transformation of M into T. The continuous transformation $fM=N$ is said to be free [H. Hopf, Fund. Math. **28**, 33–57 (1937)] provided there is a continuous transformation g of M into M such that fgx is distinct from fx for every x in M. The transformations f and g of M into N have a coincidence [S. Lefschetz, Topology, Amer. Math. Soc. Colloq. Publ., vol. 12, New York, 1930] if there is an x in M for which $fx=gx$.

D. W. Hall (Providence, R. I.).

Weinberg, N. Sur les espaces topologiques régulièrement fermés. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 523–524 (1941). [MF 4948]

A regular topological space R is "r-closed" if it is closed in every regular space containing it. The set $\{V\}$ of open sets V "r-cover" R if they cover R and if to every V there is a V' in the set such that $\bar{V}\subset V'$. The space R is "r-compact" if every r-covering contains a finite subset which covers R, and R is "strongly regular" if for each point x and neighborhood $U(x)$ of x there is a sequence $U_i(x)$ such that $\overline{U_i(x)}\subset U_{i+1}(x)\subset U(x)$, $i=1, 2, \cdots$. A system $\{G\}$ of nonempty sets is a "regular set of open sets" ["ouverture régulière," Alexandroff, Rec. Math. [Mat. Sbornik] N. S. **5** (47), 403–423 (1939); cf. these Rev. **1**, 318] if for G_1, G_2 there is a G_3 with $\bar{G}_3\subset G_1G_2$ and if $\Pi\, G=0$. Proofs are given of the theorems: (1) a regular R is r-closed if and only if it is r-compact; (2) a space which is r-closed and strongly regular is bi-compact; (3) a regular space is r-closed if and only if it has no "regular set of open sets."

W. W. Flexner (Ithaca, N. Y.).

Chevalley, Claude and Frink, Orrin, Jr. Bicompactness of cartesian products. Bull. Amer. Math. Soc. **47**, 612–614 (1941). [MF 5054]

The authors give a very simple proof of Tychonoff's theorem that the cartesian product of any number of bicompact spaces is bicompact. *H. Wallman.*

Szpilrajn, Edward. Remarque sur les produits cartésiens d'espaces topologiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 525–527 (1941). [MF 4949]

Let X^T be the cartesian product of a family of sets $X(t)$, one for each $t\epsilon T$. If each $X(t)$ is a topological space, X^T is topologized according to E. Čech [Ann. of Math. (2) **38**, 823–844 (1937)]. It is shown that, if T is non-enumerable and, for each t, $X(t)$ is a topological space having at least two points and an enumerable basis, then X^T contains a closed subset which is not a continuous image of X^T. The contrary is known to be true for the Cantor discontinuum

which is expressible as X^T for enumerable T and $X(t)$ having just two points. *L. W. Cohen* (Lexington, Ky.).

Hall, D. W. and Puckett, W. T., Jr. Strongly arcwise connected spaces. Amer. J. Math. **63**, 554–562 (1941). [MF 4678]

Recently the authors introduced strongly arcwise connected spaces as characterizing those cyclic Peano spaces A in which every arc-preserving transformation T (continuity not assumed) is a homeomorphism or $T(A)$ is an arc [cf. Bull. Amer. Math. Soc. **47**, 468–475 (1941); these Rev. **2**, 325]. A space is strongly arcwise connected if every infinite subset has an infinite intersection with some arc. The present paper proves that Peano spaces have this property if and only if for every infinite family of open sets there is an arc intersecting infinitely many of the sets. A space is defined to be strongly arcwise connected at a point p if every infinite sequence $x_i \to p$ has an infinite intersection with some arc. They show that A fails to be strongly arcwise connected at p if and only if there is a closed set N and a separation $A-N=M+\sum_1^\infty K_i$, where the K_i are components such that $K_i \to p$ and $F(K_i) \subset p+p_i \subset N$. From this it follows that a cyclicly connected Peano space is strongly arcwise connected at every point, save a countable number. *W. L. Ayres* (Lafayette, Ind.).

Alexandroff, P. und Proskuriakoff, I. Über reduzible Mengen. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 217–224 (1941). (Russian. German summary) [MF 5172]

The residue of a subset A of a space is defined, after Hausdorff, as $A_1 = \text{res}\, A = \overline{(\bar{A}-A)}-(\bar{A}-A)$; it is proved that in order that a subset A of a locally bicompact Hausdorff space shall be bicompact at the point $x \epsilon A$ it is necessary and sufficient that $x \epsilon A - \text{res}\, A$. Motivated by this, another residue is defined for sets A. Define $A^0 = A$, and, if A^β is defined for all ordinals $\beta < \alpha$, let A^α be the set of points of $A^{\alpha-1}$ (if α has a predecessor) at which $A^{\alpha-1}$ is not bicompact; if α has no predecessor, let $A^\alpha = \Pi_{\beta<\alpha} A^\beta$. If, for some α, A^α is empty and A^β is not, $\beta < \alpha$, then A is called α-reducible. This parallels the construction of the Hausdorff residues $A_0 = A$, $A_1, \cdots, A_\alpha = \text{res}\, A_{\alpha-1}$ or $\Pi_{\beta<\alpha} A_\beta$ (according as α has or not a predecessor). The set A is αH-reducible if A_α is empty, and no earlier residue is empty. It is proved that the notions α-reducible and αH-reducible coincide in locally bicompact spaces. With an appropriate modification of the definition of reducibility, it is shown that this equivalence holds also in locally compact Hausdorff spaces satisfying the first countability axiom. It follows that in spaces of these types at least, the Hausdorff reducibility is a topological invariant. Two concluding results: (1) in order that a set A in a compact metric space be simultaneously an absolute F_σ-set and absolute G_δ-set, it is necessary and sufficient that every closed subset of A contain at least one point at which it (the subset) is locally compact; (2) an absolute F_σ-set which can be mapped in a reciprocally one-one and (one-way) continuous fashion upon an absolute G_δ-set must be itself an absolute G_δ. *L. Zippin.*

Alexandroff, Paul. Der endliche dimensionstheoretische Summensatz für bikompakte Räume. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 1–6 (1941). (Russian. German summary) [MF 5291]

The addition theorem of dimension theory is proved for a finite number of summands in a bicompact Hausdorff space. This was proved for normal spaces, in the full (that is, countable) case by E. Čech [Časopis Pěst. Mat. Fys. **62**, 277–291 (1933); Czech with French summary]. In these theorems dimension is taken in the sense of order of finite coverings. It is also shown that the dimension in the sense of coverings of a bicompact Hausdorff space never exceeds its inductive, that is, Menger-Urysohn, dimension.
L. Zippin (Flushing, N. Y.).

Vedenissoff, N. Sur la dimension au sens de E. Čech. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 211–216 (1941). (Russian. French summary) [MF 5171]

If R is an arbitrary space, let us say Dim $R=-1$ if R is empty, and Dim $R \leqq n$ if for every closed set F of R and open set $U \supset F$ there is an open set V, $F \subset V \subset U$, for which Dim boundary $V \leqq n-1$. The definition of dim $R \leqq n$ is the same, except that the closed sets F are restricted to be the points of R. We say $\underline{\dim}\, R \leqq n$ if every finite covering by open sets of R has a refinement of order not greater than n. Let R be a normal space. Then βR and ωR, Čech's [Ann. of Math. (2) **38**, 823–844 (1937)] and Wallman's [Ann. of Math. (2) **39**, 112–126 (1938)] extensions of R, are identical. The main result of the author is that Dim $R=$ Dim βR if R is normal. Moreover, combining this with $\underline{\dim}\, R = \underline{\dim}\, \beta R$ [Wallman, loc. cit.] and Alexandroff's result [cf. the preceding review] that for a bicompact space dim $R \geqq \underline{\dim}\, R$, the author shows that, if R is normal, then Dim $R \geqq \underline{\dim}\, R$, thereby generalizing an inequality for perfectly normal spaces due to Čech [Bull. Int. Acad. Sci. Bohême **33**, 38–55 (1932)].
H. Wallman (Madison, Wis.).

Ehresmann, Charles et Feldbau, Jacques. Sur les propriétés d'homotopie des espaces fibrés. C. R. Acad. Sci. Paris **212**, 945–948 (1941). [MF 5037]

This note partly duplicates one by Hurewicz and Steenrod [Proc. Nat. Acad. U.S.A. **27**, 60–64 (1941); these Rev. **2**, 323], which, owing to the present circumstances, had remained unknown to the authors. The definition here used for fibre-spaces is intermediate between that of Hurewicz-Steenrod and that of Whitney [Proc. Nat. Acad. Sci. U.S.A. **26**, 148–153 (1940); these Rev. **1**, 220], as it depends upon a group of homeomorphisms of the fibre into itself; when this group is the group of all such homeomorphisms it probably becomes equivalent to that of Hurewicz-Steenrod. A "deformation lemma" is given for such spaces, which does not materially differ from Hurewicz-Steenrod's theorem 1 ("covering homotopy theorem"); it asserts that, if $\Phi_0(K)$ is a mapping of a finite complex K into fibre-space E, $\varphi_t(K)$ a deformation $(0 \leqq t \leqq 1)$ of the projection $\varphi_0(K)$ of $\Phi_0(K)$ onto the base-space B, there is a deformation $\Phi_t(K)$ of $\Phi_0(K)$ in E, the projection of which is $\varphi_t(K)$. From this the authors deduce the following results on the homotopy groups of a (connected) fibre-space E, with fibre F, over base-space B: F_0 being a fixed fibre in E, let $\pi_n'(F_0)$ be the subgroup of $\pi_n(F_0)$ consisting of those mappings of S^n into F_0 which are homotopic to 0 in E; let $\pi_n''(E)$ be the subgroup of $\pi_n(E)$ consisting of those elements which can be represented by mappings of S^n into F_0; let $\pi_n'''(B)$ be the subgroup of $\pi_n(B)$ consisting of projections of mappings of S^n into E. Then $\pi_{n-1}'(F_0)$ is isomorphic to $\pi_n(B)/\pi_n'''(B)$ for $n>1$; $\pi_n''(E)$ to $\pi_n(F_0)/\pi_n'(F_0)$ for $n \geqq 1$; and $\pi_n'''(B)$ to $\pi_n(E)/\pi_n''(E)$ for $n \geqq 1$; if F is connected, $\pi_1'''(B)$ is isomorphic to $\pi_1(B)$. The method of proof is briefly indicated.

Applications are given to complex projective spaces and to lens-spaces (spaces whose universal covering is a sphere).
A. Weil (Haverford, Pa.).

Fox, Ralph H. Extension of homeomorphisms into Euclidean and Hilbert parallelotopes. Duke Math. J. 8, 452–456 (1941).

The problem of extending a given homeomorphism is a generalization of the imbedding problem. This theorem is a generalization of the Menger-Nöbeling imbedding theorem: Let A be a compact subset of a separable metrizable space X. Let n be the dimension of $X-A$. Let y be a point of the n-dimensional parallelotope E^n. If f is a homeomorphism of A into the $(q+n)$-dimensional parallelotope $Y=E^q\times E^n$, where $q\geqq 1+\dim X$, and if $f(A)\subset E^q\times[y]$, then f can be extended to a homeomorphism of X into Y. The theorem holds also when $n=\infty$; it is then a generalization of the Urysohn imbedding theorem. In comparison with previous results of Gehman and Adkisson-MacLane for the Euclidean plane, the theorem imposes no bound on the dimension of the space, but instead requires that restrictions be placed on the homeomorphism. It is further proved that the homeomorphic extensions of a given homeomorphism constitute a residual set in the space of all extensions.
H. M. Gehman (Buffalo, N. Y.).

Alexandroff, P. Zurückführung des Alexander-Pontrjaginschen Dualitätssatzes auf den Dualitätssatz von Kolmogoroff. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 401–410 (1940). (Russian. German summary) [MF 5278]

This note establishes the following theorem: Let G be an open subset of R^n (Euclidean or spherical n-space). Let p be an integer, $0\leqq p\leqq n$. The groups $B_\nabla{}^p(G, X)$ and $B^{n-p}(G, X)$ are isomorphic. The notation follows the author's "General Combinatorial Topology" [Trans. Amer. Math. Soc. 49, 41–105 (1941); see these Rev. 2, 323]. From this theorem it follows that the duality theorem of Kolmogoroff [loc. cit.] has the classical duality theorem of Alexander, as extended by Pontrjagin, as a consequence.
L. Zippin (Flushing, N. Y.).

Levi, B. Plane polygons and Jordan's theorem. Math. Notae 1, 9–26 (1941). (Spanish) [MF 5008]

The author gives a proof that a simple closed polygon separates the plane. The interior of a convex polygon is defined as the set of points which, for every side of the polygon, lie in the same half-plane (determined by the side) as the vertices not on that side. For a general polygon, a systematic decomposition into convex polygons is set up, and the interior is defined by means of the decomposition. The theorem is then proved by induction on the number of convex polygons in the decomposition. Other results about polygons are also proved: for example, a simple closed polygon has at least three angles which are not re-entrant; if four points are taken in order on a simple closed polygon P, and the first and third are on another polygon Q interior to P, then any polygon interior to P and joining the second and fourth points must intersect Q.
R. P. Boas, Jr.

Jones, F. Burton. Certain consequences of the Jordan curve theorem. Amer. J. Math. 63, 531–544 (1941). [MF 4676]

Since relatively little is known of the nature of locally connected spaces in which the Jordan curve theorem holds, except for cases where properties such as local compactness are assumed, the author undertakes an investigation of the case where the space is a complete Moore space. More specifically, if S is a space satisfying Axioms 0–4 of R. L. Moore's "Foundations of Point Set Theory" [Amer. Math. Soc. Colloquium Publ., vol. 13, New York, 1932], then S will be similar to certain subsets of the Euclidean plane. For example, a modified Janiszewski theorem holds: If H and K are two continua such that $H\cdot K$ is not connected and the boundary of $H\cdot K$ rel. K is a subset of a compact open subset of K, then $H+K$ separates S.
R. L. Wilder.

Jones, F. Burton. Aposyndetic continua and certain boundary problems. Amer. J. Math. 63, 545–553 (1941). [MF 4677]

A continuum M is "aposyndetic" if for every $x, y\epsilon M$, $x\neq y$, there exists a continuum K in $M-y$ such that x is interior to K. In a Hausdorff space the class of aposyndetic continua includes both the connected im kleinen and the semi-locally-connected continua. However, if the aposyndetic continuum M is locally peripherally bicompact, then M is semi-locally-connected. Moreover, the aposyndetic continua are identified with the "freely decomposable" continua; a continuum M is freely decomposable if, for every $x, y\epsilon M$, M is the sum of two continua neither of which contains both x and y. The latter result characterizing aposyndetic continua is comparable with a characterization of locally peripherally bicompact continuous curves which is obtained by replacing "point" y above by "continuum" y. In a final section on "boundary problems" it is shown that in a space satisfying Axioms 0–4 of R. L. Moore's "Foundations of Point Set Theory" [Amer. Math. Soc. Colloquium Publ., vol. 13, New York, 1932] every component of the boundary of a complementary domain of a locally compact aposyndetic continuum is a locally compact continuous curve.
R. L. Wilder (Ann Arbor, Mich.).

Jones, F. B. Monotonic collections of peripherally separable connected domains. Bull. Amer. Math. Soc. 47, 661–664 (1941). [MF 5062]

A collection G of point sets is called monotonic if $g, g'\epsilon G$ implies $g\supset g'$ or $g'\supset g$. A subcollection H of a collection G of point sets is said to run upward through G if for each $g\epsilon G$ there exists $h\epsilon H$ such that $g\subset h$. Let S be a locally connected metric space. The principal theorem is to the effect that, if G is a monotonic collection of peripherally separable connected domains of S, then some countable subcollection of G runs upward through G. As corollaries one has that in S every well-ordered increasing sequence of peripherally separable connected domains is countable, and the theorem proved earlier by the author [Bull. Amer. Math. Soc. 41, 437–439 (1935)] that a connected space of type S which is locally peripherally separable is completely (perfectly) separable.
R. L. Wilder (Ann Arbor, Mich.).

Wojdysławski, M. Sur les rétractes par déformation des coupures de la surface sphérique. Studia Math. 9, 166–180 (1940). (French. Ukrainian summary) [MF 5265]

Let X be an arbitrary subset of the plane. The author proves that in order that X be an absolute neighborhood retract [in the generalized sense of C. Kuratowski, Fund. Math. 24, 269–287 (1935)] it is necessary and sufficient that (1) X be locally arcwise connected, (2) if a point p is a limit of a sequence of components of the complement of X, then p is not in X. In order to get the similar theorem for absolute retracts, (2) is to be replaced by (2′) the complement of X is connected.
S. Eilenberg (Ann Arbor, Mich.).

Choquet, Gustave. Points invariants et structure des continus. C. R. Acad. Sci. Paris **212**, 376–379 (1941). [MF 4909]

Let T be a single valued continuous transformation of a plane continuum C into itself. This note studies various kinds of such transformations and is concerned primarily with fixed points. If C is a circle and T is a homeomorphism of index $(+1)$ the set of fixed points may be any closed non-empty set; if the index of T is (-1) the set of fixed points must be a closed subset of an arc having its end points on the boundary of the circle, and it can not be anything else. If C is any continuum and T is an extendable homeomorphism of period $n\neq 2$ taking C into itself, then T has at most one fixed point, and at least one if C does not separate the plane. There are other fixed point theorems. By means of contours a classification of continua is made and the study is carried out for sets of the type thus defined. Two curious closed sets are given as examples. *D. Montgomery.*

Miranda, Carlo. Un'osservazione su un teorema di Brouwer. Boll. Un. Mat. Ital. (2) **3**, 5–7 (1940). [MF 4987]

An elementary proof of the equivalence of Brouwer's fixed point theorem with a special case of Kronecker's index theorem [Alexandroff-Hopf, Topologie, p. 467, Berlin, 1935]. *R. H. Fox* (Urbana, Ill.).

Kakutani, Shizuo. A generalization of Brouwer's fixed point theorem. Duke Math. J. **8**, 457–459 (1941).

Let S be a closed r-dimensional simplex, $\mathfrak{R}(S)$ the family of closed convex subsets of S, and Φ a point-to-set mapping $S\rightarrow\mathfrak{R}(S)$ which is upper semi-continuous in the sense that $x_n\rightarrow x_0$, $y_n\rightarrow y_0$, $y_n\epsilon\Phi(x_n)$ imply $y_0\epsilon\Phi(x_0)$. Theorem: $x\epsilon\Phi(x)$ for at least one x. *P. A. Smith* (New York, N. Y.).

Odle, John W. Non-alternating and non-separating transformations modulo a family of sets. Duke Math. J. **8**, 256–268 (1941).

Broadening concepts introduced by E. P. Vance, which in turn are extensions of the notions of non-alternating and non-separating transformations as used by the reviewer and J. F. Wardwell, respectively, the author considers an arbitrary family of sets G and a continuous mapping $T(A)=B$, where A is a compact metric continuum, and defines T to be: (1) non-separating modulo G if for no $x\epsilon B$ does $T^{-1}(x)$ separate two points of A unless a subset of $T^{-1}(x)$ belonging to G also separates these points in A; (2) non-alternating modulo G if for x, $y\epsilon B$, $T^{-1}(x)$ does not separate two points of $T^{-1}(y)$ in A unless some subset of $T^{-1}(x)$ belonging to G also separates them in A. Other notions are broadened similarly and a study is made and results obtained in this general situation analogous to that of the earlier writers in more restricted spheres. Also some new results on locally non-separating and locally non-alternating transformations are given. In particular, it is shown that 0-regularity of T as defined by A. D. Wallace implies that T is locally non-alternating. *G. T. Whyburn* (Charlottesville, Va.).

Eilenberg, Samuel. On spherical cycles. Bull. Amer. Math. Soc. **47**, 432–434 (1941). [MF 4533]

Several propositions on spherical cycles of a space X, that is, n-cycles obtained by continuous mappings of an n-sphere in X. The author shows among other results that, if M^r is an r-dimensional manifold and P^{r-n-1} an at most $(r-n-1)$-dimensional subpolytope of M^r, then every n-dimensional cycle of M^r-P^{r-n-1} which bounds in M^r is a spherical cycle with respect to M^r-P^{r-n-1}. *W. Hurewicz.*

Pontrjagin, L. Products in complexes. Rec. Math. [Mat. Sbornik] N.S. **9**(51), 321–330 (1941). (English. Russian summary) [MF 4557]

The author develops the cohomology ring of a simplicial complex by imbedding the complex in a Euclidean space and correlating the multiplication of cochains in the complex with the intersection of certain chains in the complementary space. Invariance is established by intrinsic methods which show that a product theory of the sort envisaged is essentially unique [cf. Whitney, Ann. of Math. (2) **39**, 397–432 (1938)]. *A. W. Tucker* (Princeton, N. J.).

Pontrjagin, L. A classification of mappings of the three-dimensional complex into the two-dimensional sphere. Rec. Math. [Mat. Sbornik] N.S. **9**(51), 331–363 (1941). (English. Russian summary) [MF 4558]

The following results on the number of homotopy classes of maps of an n-complex K^n in a 2-sphere S^2 are proved. Those classes of maps which are algebraically inessential are in 1-1 correspondence with the homotopy classes of maps of K^n in S^3. The correspondence is the natural one determined by the Hopf map $S^3\rightarrow S^2$. A map f of K^n in S^2 has attached to it a 2-cocycle $z^2(f)$ in K^n. A map g is homotopic to a map g' coinciding with f on the 2-dimensional part K^2 of K^n if and only if $z^2(g)\sim z^2(f)$. If f and g coincide on K^2, they determine a 3-cocycle $z^3(f, g)$ in K^n. The existence of a 1-cocycle x^1 such that $z^3(f,g)\sim 2x^1\times z^2(f)$ is a necessary and sufficient condition that g be homotopic to a map g' coinciding with f on K^3. For $n=3$, these results give a complete classification. A map f of the part K^3 of a K^4 in S^2 extends to a map of K^4 if and only if the cocycle $z^2(f)$ in K^3 is also a cocycle in K^4 and $z^2(f)\times z^2(f)\sim 0$ in K^4. These results are reinterpreted in terms of homology for $K^3=$ manifold. It is not clearly stated, but it appears that the coefficients of $z^2(f)$ and x^1 are in the homotopy group $\pi_2(S^2)$, those of $z^3(f, g)$ are in $\pi_3(S^2)$, and the multiplication of elements in $\pi_2(S^2)$ with product in $\pi_3(S^2)$, as demanded by the multiplication of cocycles, is as defined by J. H. C. Whitehead [Ann. of Math. (2) **42**, 409–428 (1941); cf. these Rev. **2**, 323]. *N. E. Steenrod* (Chicago, Ill.).

Whitehead, J. H. C. Note on manifolds. Quart. J. Math., Oxford Ser. **12**, 26–29 (1941). [MF 4668]

Examples by Cairns and van Kampen [Cairns, Ann. of Math. (2) **41**, 792–795 (1940); these Rev. **2**, 74] show that the star of a vertex in a combinatorial n-manifold [Newman, Nederl. Akad. Wetensch., Proc. **29**, 611–626 (1926); Alexander, Ann. of Math. (2) **31**, 292–320 (1930)] need not be rectilinearly imbeddable in Euclidean n-space R^n. The author shows that a combinatorial n-manifold can be so subdivided (by elementary subdivisions) that the star of any of its simplices can be rectilinearly imbedded in R^n. This implies a negative answer to a question raised by Cairns [loc. cit., p. 793]. *R. H. Fox* (Urbana, Ill.).

Bockstein, M. Über die Homologiegruppen der Vereinigung zweier Komplexe. Rec. Math. [Mat. Sbornik] N.S. **9**(51), 365–376 (1941). (German. Russian summary) [MF 4559]

This paper sharpens the Mayer-Vietoris results regarding the sum of two complexes. Suppose one knows for all m the homology groups modulo m of the two complexes and of the subcomplex they have in common, as well as the natural homomorphisms of the former groups into the latter. Then, the author shows, the homology theory of the sum of the two complexes is fully determined. *A. W. Tucker.*

Flexner, William W. Non-commutative chains and the Poincaré group. Duke Math. J. 8, 497–505 (1941).

For an abstract system S of cells, satisfying suitable conditions, a group π is defined. If S is a geometric complex, π is proved to be the Poincaré group. It is required of S that the "boundary" of a 1-cell shall be a pair of 0-cells, and the boundary of a 2-cell shall be a "closed path" of 1-cells. A notion of "subdivision" of S is defined, and π is shown to be invariant under this operation. *N. E. Steenrod.*

Hopf, Heinz. Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen. Ann. of Math. (2) 42, 22–52 (1941). [MF 3672]

The author calls a compact manifold M a "Γ-manifold" if it is possible to assign to each ordered pair (x, y) of points of M a third point z called the product $x\cdot y$ in such a way that (1) $x\cdot y$ is a continuous function of the pair (x, y) and (2) for a fixed x_0 the correspondences $x\rightarrow x\cdot x_0$ and $x\rightarrow x_0\cdot x$ are mappings (of M on M) of degrees different from 0 (it is clear that these degrees do not depend on the choice of x_0). Every group manifold is of course a Γ-manifold. Furthermore, any sphere S_n of an odd dimension is easily proved to be a Γ-manifold (while no sphere of an even dimension is a Γ-manifold). The main purpose of the present paper is the proof of the following striking result: If M is a Γ-manifold the homology ring of M (with rational numbers as coefficients) is isomorphic to the homology ring of the product space of spheres $S_{n_1}\times S_{n_2}\times\cdots\times S_{n_k}$, where the n_i are odd numbers. Consequently the Betti numbers of M are the coefficients of the polynomial $\Pi_{i=1}^{k}(1+z^{n_i})$. In other words, the polynomial $\Pi_{i=1}^{k}(1+z^{n_i})$ is the Poincaré polynomial of the manifold M.

This theorem is not only a far-reaching generalization of the well-known results of Pontrjagin about Lie groups, but in addition it throws an entirely new light on these results by emphasizing their elementary topological nature and their independence of analytic or algebraic assumptions (not even the associative law of multiplication is needed). The proof is surprisingly simple and is based upon the known fact that a continuous mapping of a compact manifold M into another compact manifold M' induces a homomorphism of the homology ring of M' into the homology ring of M (this is the so-called "Hopfs Umkehrhomomorphismus"). Now the product operation defined in a Γ-manifold M can obviously be regarded as a mapping of the product-space $M\times M$ into M; hence we have a homomorphism Φ of the homology ring of M into that of $M\times M$. The latter homology ring is of course determined by the first, and from the assumption (2) about the product operation one derives certain algebraic properties of the homomorphism Φ which as the author shows cannot be fulfilled unless the homology ring of M has the structure specified above.

[The reviewer wishes to remark that, by using the technique of cohomologies (instead of homologies), it would be easy to prove the author's result by essentially the same methods for the more general case of compact spaces M which are not necessarily manifolds, assuming that the cohomology ring of M has a finite basis and that M admits a continuous product-operation satisfying the following condition: (2′) For a fixed point x_0 the endomorphisms of the cohomology ring of M induced by the transformations $x\rightarrow x\cdot x_0$ and $x\rightarrow x_0\cdot x$ are automorphisms; if M is a manifold this condition is easily shown to be equivalent to the previous condition (2).] *W. Hurewicz* (Chapel Hill, N. C.).

Hopf, Heinz. Ein topologischer Beitrag zur reellen Algebra. Comment. Math. Helv. 13, 219–239 (1941).

This elementary paper gives a neat example of the usefulness and scope of the (essentially elementary) topological method of the "Umkehrhomomorphismus," that is, the homomorphism of the cohomology ring of a complex (or of a compact space) B into that of a complex (or compact space) A, which is induced by a mapping of A into B, a method originally due to the author [H. Hopf, J. Reine Angew. Math. 163, 71–88 (1930)]. Denoting by $-x$ the "antipode" of point x on the sphere S^{r-1}, the main result is as follows: if there is a mapping $f(x, y)$ of the product $S^{r-1}\times S^{s-1}$ into S^{n-1}, satisfying the conditions $f(-x, y) = f(x, -y) = -f(x, y)$, then the congruence $(x+y)^n\equiv 0$ (mod $2, x^r, y^s$) holds in the ring of polynomials in x, y with integral coefficients. This, when r and s are given, gives a lower bound for possible values of n, from which, as the author shows, various topological and algebraical results (some new, some due to Stiefel, Borsuk, Hurwitz, etc.) can be deduced as special cases. The proof is obtained by identification of antipodic points on the three spheres: f thus becomes a mapping of $P^{r-1}\times P^{s-1}$ into P^{n-1}, with the property that (denoting by p, p' points in P^{r-1}, P^{s-1}; by d, d', d'' straight lines in P^{r-1}, P^{s-1}, P^{n-1}) the homologies $F(p\times d') \sim F(d\times p')\sim d''$ hold mod 2. The result follows immediately by applying to this mapping the most elementary properties of the Umkehrhomomorphismus (the cohomology rings being taken mod 2). *A. Weil* (Haverford, Pa.).

Stiefel, Eduard. Über Richtungsfelder in den projektiven Räumen und einen Satz aus der reellen Algebra. Comment. Math. Helv. 13, 201–218 (1941).

Using the results of a previous paper [Comment. Math. Helv. 8, 305–351 (1936)], the author proves that if $n+1=2^\lambda\cdot u$ and u is odd then it is impossible to have on the projective n-space 2^λ continuous vector fields linearly independent at every point. One of the applications: let $n=2^\lambda\cdot u$, u odd, and let real square n-row matrices $A_1, A_2, \cdots, A_s$ be given such that $y_1A_1+y_2A_2+\cdots+y_sA_s$ is non-singular except for $y_1=y_2=\cdots=y_s=0$; then $s\leqq 2^\lambda$. *S. Eilenberg.*

Bachiller, T. R. Comments on algebra and topology. Revista Mat. Hisp.-Amer. (4) 1, 68–74 (1941). (Spanish) [MF 5096]

A short survey of some fundamental concepts in topology, leading up to the following result: the nth homotopy group of a topological product is the direct product of the nth homotopy groups of the factors. *A. Weil.*

Elsholz, L. Zur Theorie der Änderung der topologischen Invarianten der Niveauflächen. Rec. Math. [Mat. Sbornik] N.S. 8(50), 463–470 (1940). (Russian. German summary) [MF 3743]

Let M^n be an n-manifold and $\bar{n}(A)$ an integer-valued function of a closed set $A\subset M^n$ satisfying the following conditions: (1) $\bar{n}(A)=1$ if A is a single point; (2) $\bar{n}(A)=\bar{n}(B)$ if A is isotopic to B in M^n; (3) $\bar{n}(A+B)\leqq\bar{n}(A)+\bar{n}(B)$; (4) $\bar{n}(U(A, \epsilon))=\bar{n}(A)$, where $U(A, \epsilon)$ is the ϵ-neighborhood of A and $\epsilon>0$ sufficiently small; (5) $A\subset B$ implies $\bar{n}(A)\leqq\bar{n}(B)$. Lusternik-Schnirelman category, homology category and several other known invariants are examples of functions of this type. The author applies the functions $\bar{n}(A)$ to study topological properties of the "level sets"

$(f=x)$, f being a twice differentiable real valued function defined on the manifold M. He shows that $\bar{n}(f=x)$ cannot change more than one unity when x crosses a critical value whose critical set consists of a finite number of points. After having imposed on the function $\bar{n}(A)$ the additional condition: (6) $\bar{n}(A+B)=\max(\bar{n}(A), \bar{n}(B))$ if $AB=0$ (this condition is satisfied in case of category), several other propositions are established, for example: $\max \bar{n}(f=x) \geqq [\bar{n}(M^n)/2]$. (Under certain conditions it can be shown that $\bar{n}(f=x)$ increases monotonously until this maximum value is reached and decreases monotonously after that.) *W. Hurewicz.*

Threlfall, W. Stationäre Punkte auf geschlossenen Mannigfaltigkeiten. Jber. Deutsch. Math. Verein. **51**, 14–33 (1941).

The present paper is an account of a lecture on the theory of critical points on closed manifolds. The paper is descriptive in character and, with the help of examples, the critical point relations of Morse are obtained in an intuitive fashion for the nondegenerate case. Modifications needed in degenerate cases are described. It is also indicated how the concept of category can be used in order to determine the minimum number of critical points. *M. R. Hestenes.*

RELATIVITY

***Einstein, A., Bargmann, V. and Bergmann, P. G. On the five-dimensional representation of gravitation and electricity.** Theodore von Kármán Anniversary Volume, pp. 212–225. California Institute of Technology, Pasadena, Calif., 1941.

The authors show that the integro-differential field equations previously obtained by two of the authors [Ann. of Math. (2) **39**, 683 (1938)] for variables defined in a five dimensional Riemannian of a particular structure may be replaced by a system of differential equations. However, these equations are not suitable for a physical theory since they offer no explanation for the empirical fact that the electrostatic force between two particles is so much stronger than the gravitational force. *A. H. Taub.*

Potier, Robert. Sur les équations de la gravitation. C. R. Acad. Sci. Paris **212**, 295–298 (1941). [MF 4900]

It is postulated that (1) the world lines of material particles are geodesics and (2) the Ricci tensor $R^i{}_j$ is invariant under three dimensional space-like rotations. The relation between these postulates and the field equations of general relativity is examined. *A. H. Taub.*

Lorenz, Edward N. A generalization of the Dirac equations. Proc. Nat. Acad. Sci. U. S. A. **27**, 317–322 (1941). [MF 4582]

The generalization referred to in the title is obtained by generalizing the matrices A_m which occur in the Dirac equation

$$(1/\lambda)\sum_{m=1}^{4} A_m\partial\psi/\partial x_m-\psi=0,$$

and which satisfy

$$(*) \qquad A_mA_n+A_nA_m=2\delta_{mn}I, \qquad n, m=1, 2, 3, 4.$$

It is first remarked that if a fifth matrix $A_0=iI$ is introduced and if

$$A^*=\|a_{ij}^*\|=\sum_{m=0}^{4} A_my_m,$$

where $y_0, \cdots, y_4$ form a set of independent variables, then

$$\text{(I)} \qquad |A^*|=Q^2=(y_0^2+y_1^2+y_2^2+y_3^2+y_4^2)^2.$$

(II) The cofactor of a_{ij}^* in $|A^*|$ is equal to A_{ij}^*Q for some linear function A_{ij}^* of $y_0, y_1, \cdots, y_4$. The relation between (I), (II) and (*) is then examined. A more detailed discussion of this relationship which includes the geometric as well as algebraic relations may be found in the lectures of W. Givens [published in notes of a seminar conducted by O. Veblen and J. von Neumann, Geometry of Complex Domains, Princeton Mimeographed Notes 1935–1936, p. 6–10]. The generalized matrices A_m are taken to be those satisfying (I). The general solution of (I) for (4×4) matrices is not given. However, a particular solution is given. It is of the form $\sum_{m=0}^{4} y_m\alpha_m$, where $\alpha_m=A_m+t_mR$, $\alpha_0=I+t_0R$, where t_m is arbitrary and R is a 4×4 matrix of rank one such that $\sum_{m=0}^{4} y_m\alpha_m$ satisfies (I). The proposed generalization of the Dirac equation is

$$(1/\lambda)\sum_{m=1}^{4}(A_m+t_mR)\partial\psi/\partial x_m-(I+t_0R)\psi=0.$$

It is pointed out that the solutions of this equation are not solutions of (†) $(\Box-I)\psi=0$, where

$$\Box=\frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}\right),$$

but instead have the property that

$$(\Box-I)(\Box-(I+t_0R))\psi=0.$$

Since (†) is not satisfied by the solutions of the generalized equation the relations between the physical consequences of this equation and those of non-relativistic quantum mechanics will be different from those between the latter and the Dirac equation. *A. H. Taub* (Seattle, Wash.).

Haantjes, J. The conformal Dirac equation. Nederl. Akad. Wetensch., Proc. **44**, 324–332 (1941). [MF 5015]

The behavior of the Dirac equation under conformal transformations $g_{ik}\to\sigma^2g_{ik}$ is examined. The following known results are obtained [cf. Pauli, Helvetica Phys. Acta **13**, 204–208 (1940); these Rev. **2**, 144]: (1) The Dirac equation is conformally invariant for particles of zero mass. (2) If it is assumed that the mass becomes multiplied by σ^{-1} then the Dirac equation for particles with non-vanishing mass is conformally invariant. Some physical considerations of this assumption are given. Among them is one in which the probability function is taken to be

$$(*) \qquad \omega_{\bar{A}B}\bar{\psi}^{\bar{A}}\psi^B,$$

where ψ^A is the solution of the Dirac equation, $\omega=\|\omega_{\bar{A}B}\|$ is the fundamental Hermitian spinor satisfying $\omega\alpha^h=\bar{\alpha}^h\omega$ $(h=1, 2, 3, 4)$, and α^h are the fundamental spin matrices. Since the Hermitian form (*) is not positive definite it would seem that the customary interpretation of the probability function as the time component of the current vector is more acceptable. *A. H. Taub* (Seattle, Wash.).

Lichnerowicz, André. Sur la définition géométrique des processus matériels en relativité générale. C. R. Acad. Sci. Paris **212**, 421–423 (1941). [MF 4913]

Let $T_{\lambda\mu}$ be a symmetric energy tensor in space-time; the determinantal equation $|T_{\lambda\mu}+sg_{\lambda\mu}|=0$ in general defines four invariants, of which one corresponds to density and the other three to stresses in the medium. The author is interested in the case where there is a double root, that is, a

partially symmetric stress. He states that in this case it is possible to write the energy tensor in the form

$$T_{\lambda\mu}=(\sigma+\pi)u_\lambda u_\mu-\pi g_{\lambda\mu}+\tau_{\lambda\mu},$$

where $\tau_{\lambda\mu}$ is the energy tensor of an electromagnetic field. This expression may be said to involve three invariants; of these σ and π are two, while the third is given by $|\tau_{\lambda\mu}+sg_{\lambda\mu}|=0$, it being well known that the roots of this equation are of the form $K, K, -K, -K$. This energy tensor is to be applied to the theory of the electron, but only very brief indications of results are given; they are to be developed in a later paper. *J. L. Synge* (Toronto, Ont.).

Lichnerowicz, André. Les espaces à connexion semi-symétrique et la mécanique. C. R. Acad. Sci. Paris **212**, 328–331 (1941). [MF 4906]

A connection is defined which appears as a modification of that of the space introduced by Weyl. Mention is made of the possibility of application to relativity and mechanics. A detailed treatment is promised in a paper under preparation. *T. Y. Thomas* (Los Angeles, Calif.).

Lichnerowicz, André et Marrot, Raymond. Propriétés statistiques des ensembles de particules en relativité restreinte. C. R. Acad. Sci. Paris **210**, 759–761 (1940). [MF 4889]

This note extends in an obvious manner to the framework of restricted relativity the classical integro-differential equation of Boltzmann governing the effect of collisions in changing the distribution of particles in phase-space.
B. O. Koopman (New York, N. Y.).

Järnefelt, Gustaf. Note on the mass-particle in an expanding universe. Ark. Mat. Astr. Fys. **27 A**, no. 15, 10 pp. (1941). [MF 4355]

The author discusses in detail the orbit of a test particle in a field whose line element may be obtained from the "isotropic" form of Schwarzschild's solution of the field equations with cosmological constant $\lambda=0$ [A. S. Eddington, The Mathematical Theory of Relativity, Cambridge, England, 1923, p. 93] on replacing the mass m by me^{-kt}. [This is therefore equivalent to a discussion of the geodesics of the static form of the Schwarzschild solution for cosmological constant $\lambda=3k^2$, for the reviewer has shown [Trans. Amer. Math. Soc. **29**, 493 (1925)] that either of these line elements may be thrown into the other by a coordinate transformation.] *H. P. Robertson* (Princeton, N. J.).

Sakuma, Kiyosi. Cosmology in terms of wave geometry. VII. Some characteristics of the universe. J. Sci. Hirosima Univ. Ser. A. **11**, 15–20 (1941). [MF 4883]

The physical interpretations of the vector $u^i=\alpha\psi^+A\gamma^i\psi$ [these Rev. **2**, 208] are reexamined. It is further shown that on the assumption $N=\alpha\psi^+A\gamma_5\psi=0$ the scalar α may be chosen so that U^i satisfies a continuity equation.
A. H. Taub (Seattle, Wash.).

Sibata, Takasi. Cosmology in terms of wave geometry. VIII. Observation systems in cosmology. J. Sci. Hirosima Univ. Ser. A. **11**, 21–45 (1941). [MF 4884]

The author shows that the fundamental equation of wave geometry used in the wave geometry cosmology theory [these Rev. **2**, 208] is invariant under the ten parameter group of motions in de Sitter space provided the functions ψ undergo an appropriate transformation. *A. H. Taub.*

Iwatsuki, Toranosuke and Sibata, Takasi. Cosmology in terms of wave geometry. IX. Theory of spiral nebulae. J. Sci. Hirosima Univ. Ser. A. **11**, 47–88 (1941). [MF 4885]

A theory of spiral nebulae is attempted on the basis of cosmology in terms of wave geometry. The fundamental equation is restricted to be invariant under a four parameter subgroup of the group of motions of de Sitter space. There are two unequivalent such equations and each leads to a velocity vector. The velocity vector for the nebulae is taken to be a linear combination of these. General properties of spiral nebulae are examined on the basis of these and further assumptions. *A. H. Taub* (Seattle, Wash.).

MATHEMATICAL PHYSICS

Møller, C. and Rosenfeld, L. On the field theory of nuclear forces. Danske Vid. Selsk. Math.-Fys. Medd. **17**, no. 8, 72 pp. (1940). [MF 4880]

The authors show how to separate a "static" part in the interaction between nucleons (protons and neutrons) due to their mesonfield. This static part is analogous to the Coulomb interaction of electrically charged particles. In the currently used Meson field theories the static part contains, apart from the well-known Yukawa-potential, a term similar to a dipole field which diverges cubically and does not admit stationary states of the deuteron unless arbitrarily cut off. The authors show, however, that a suitable combination of a vector mesonfield with a pseudoscalar field gives rise to a non-divergent static field which is radially symmetric. The dynamic parts of the interaction, which are obtained in higher approximation, have dipole character and are supposed to cause the quadrupole moment of the deuteron. The authors give a quantitative discussion of the results of the meson interaction proposed by them.
V. Weisskopf (Rochester, N. Y.).

Møller, C. On the theory of mesons. Danske Vid. Selsk. Math.-Fys. Medd. **18**, no. 6, 46 pp. (1941). [MF 4504]

This paper is an elaboration of an idea put forward by Møller and Rosenfeld [cf. the preceding review]. The field of the nuclear forces is assumed to consist of a suitable combination of a vector field of mesons and of a pseudoscalar field. In the present paper it is shown that this combination can be derived in a less arbitrary way by postulating some sort of invariance in a five-dimensional representation. It can be interpreted as an invariance with respect to the group of space and time rotations and translations in a de Sitter space of general relativity. The combination so obtained has static solutions free of fields with dipole character. The angular dependence of the field and hence the quadrupole moment of the duteron may be obtained in higher approximations by evaluating the non-static terms. A qualitative argument shows that the order of magnitude is sufficient. *V. Weisskopf* (Rochester, N. Y.).

Kramers, H. A. and Wannier, G. H. Statistics of the two-dimensional ferromagnet. I. Phys. Rev. (2) **60**, 252–262 (1941). [MF 5025]

A new method is developed for the statistical treatment of one- and two-dimensional models for ferromagnetism (a regular array of spins, each capable of two orientations, with an interaction energy dependent only on the relative orientation of neighbors). The probability of orientation $P(n)$ of an added spin is expressed in terms of the probability function $P(n-1)$ of the preceding one. Since, for large n, $P(n)$

approaches $P(n-1)$, a functional equation for $P(n)$ is obtained which can be brought into the form of a characteristic value problem for a certain matrix. It is shown that the partition function for the ensemble is given directly by the largest characteristic value of this matrix. The linear case can be solved explicity with known results (no ferromagnetism). For the two-dimensional case, involving infinite matrices, no complete solution has been obtained. It is, however, possible to deduce from symmetry arguments that a single Curie point. if it exists, is only possible for $J/kT=0.8814$, where J is the coupling energy between neighboring spins. It can be shown, furthermore, that the specific heat at the Curie point is either infinite or else that both the specific heat and the total energy have to be continuous there. *L. W. Nordheim* (Durham, N. C.).

Kramers, H. A. and Wannier, G. H. Statistics of the two-dimensional ferromagnet. II. Phys. Rev. (2) **60**, 263–276 (1941). [MF 5026]

The matrices used in I [cf. the preceding review] for the treatment of models for ferromagnetism are of finite order if strips of finite width are considered in place of a complete two-dimensional array. The solution of the full problem can then be approximated by taking strips of increasing width, which are treated numerically. It is made plausible by this method that the specific heat at the Curie point will become infinite. A closed form approximation is then obtained by a variational method. It is found by comparing the power series at very high and at very low temperatures with the exact developments of the partition function and with those obtained from other approximations (Heisenberg, Kirkwood, Bethe) that the new method constitutes a considerable improvement compared to the earlier procedures.
L. W. Nordheim (Durham, N. C.).

***Infeld, Leopold. On the Theory of Brownian Motion.** University of Toronto Studies, Applied Mathematics Series, no. 4. University of Toronto Press, Toronto, Ont., 1940. 42 pp. $1.25.

An exposition of two theories of Brownian motion based on those of Smoluchowski and Langevin, their comparison and critique. It presupposes merely advanced calculus, and little technical knowledge of statistical mechanics; nevertheless a considerable maturity in physical thought is required if the innumerable ad hoc assumptions which are made are to appear reasonable. As in most present day expositions of this class of subject, the logical connection of the various ideas is by no means always satisfactory, as the author is fully aware. *B. O. Koopman* (New York, N. Y.).

Burgers, J. M. On the distinction between irregular and systematic motion in diffusion problems. Nederl. Akad. Wetensch., Proc. **44**, 344–353 (1941). [MF 5016]

Remarks are made first on the detection of systematic motions by the formation of mean values. A mean over a time T for a single particle and a mean of this last mean over N particles are among those considered.

To arrive at a distinction between systematic motion and diffusion it is assumed that the distribution function $f(\xi, l, u)$ can be expanded in the form

$$f(x-l, l, u)=f(x, l, u)-l\partial f(x, l, u)/\partial x.$$

The general diffusion equation which is obtained is of type

$$q=n(u_s-\bar{l}\partial u_s/\partial x)-\partial(n\cdot\overline{u'\cdot l})/\partial x,$$

where n is the number of particles per unit volume. At time t in the element Ω (between x and $x+dx$), $q=q(x, t)$ is the mean resultant current of particles through the element Ω at time t, u_s is the systematic motion and u' is the departure from this motion. The quantity $\overline{ul}=u_s\bar{l}+\overline{u'l}$ is the mean value of the product ul at time t, where in the interval from $t-T$ up to t the particles had moved over distances lying between l and $l+dl$. The hypotheses are carefully discussed. *H. Bateman* (Pasadena, Calif.).

Nagamiya, Takeo. Statistical mechanics of one-dimensional substances. II. Proc. Phys.-Math. Soc. Japan (3) **22**, 1034–1047 (1940). [MF 3798]

This paper continues a previous study by the same author of one-dimensional assemblies of molecules [Proc. Phys.-Math. Soc. Japan (3) **22**, 705–720 (1940); these Rev. **2**, 139]. Close to the particle the potential of interaction is ∞; it is linearly increasing in a further zone, constant outside. The usual mathematical mechanisms of the classical kinetic theory are introduced, a detailed computation of the partition function, etc. The author sought to find a discontinuity in the passage from the "gaseous" to the "liquid" state, corresponding to condensation. The result was negative, perfect continuity existing. This very fact, and the mathematical manipulations which are carried out in great detail, appeared to the author to constitute the interest of the paper. *B. O. Koopman* (New York, N. Y.).

Leontovich, M. Relaxation in liquids and scattering of light. Acad. Sci. USSR. J. Phys. **4**, 499–514 (1941). [MF 5197]

The theory of relaxation in a liquid consisting of anisotropic molecules is developed in terms of the deformation tensor e_{ik} and an anisotropic tensor ξ_{ik} whose diagonal sum $\xi=\xi_\alpha^\alpha$ is supposed to vanish. The free energy ψ of a unit volume of the liquid is expressed as a quadratic function of the components of e_{ik} and ξ_{ik} and the state of internal equilibrium is found by making ψ a minimum under the condition $\xi=0$. The resulting equation may be expressed in the form $\zeta_{ik}=0$, where $\zeta_{ik}=e_{ik}-\frac{1}{3}e\delta_{ik}-\xi_{ik}$ and ψ takes the form $2\psi=\kappa e^2+2\mu\zeta_\alpha^\beta\zeta_\beta^\alpha$. In a small disturbance of the equilibrium the equations of motion and of change of anisotropy are $\rho\dot{u}_i=\partial S_{i\alpha}/\partial x_\alpha$, where $S_{ik}=\kappa e\delta_{ik}+2\mu\zeta_{ik}$, and $\tau\dot{\xi}_{ik}=\zeta_{ik}$, where τ is the relaxation time which is related to the viscosity η by the equation $\eta=\mu\tau$ and to Debye's relaxation time τ_D by the equation $3\tau=\tau_D$. It is stated that τ also coincides with the relaxation time of the Kerr effect. When $\zeta_{ik}\neq0$ there is a double refraction determined by an optical dielectric tensor $\Delta\epsilon_{ik}$ equal to $A\zeta_{ik}$, where A is a constant connected with the constant α, which determines the double refraction due to flow, by the relation $\alpha=A\tau$. The depolarization of the total molecular scattered light is also connected with this optical anisotropy due to fluctuations and is studied here by a statistical method of averaging. The scattering of light in liquids with relaxation is also studied and the structure of the Rayleigh line is investigated. In an appendix the mean squares of the Fourier coefficients are calculated for accidental processes. *H. Bateman.*